Discussion session on Stochastic processes

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Outlook

A. Classical

- Anomalous diffusion in biological systems
- Kinetic Ising Models

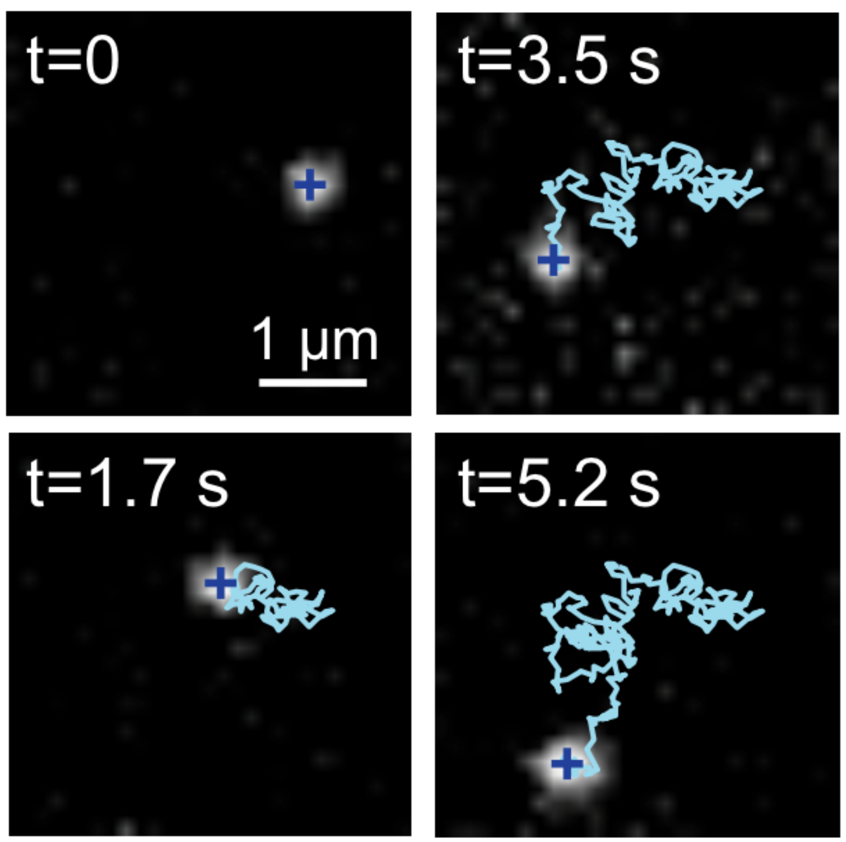
B. Quantum

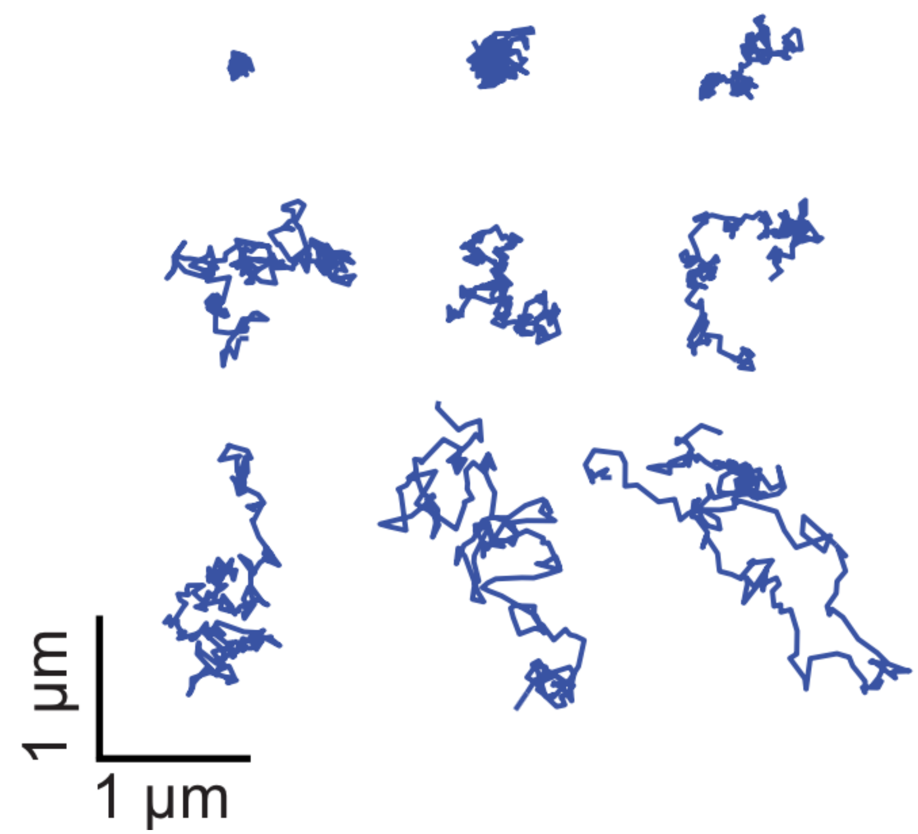
- Quantum Kinetic Ising Models
- Quantum Brownian Motion

Diffusion in a complex medium

- Single particle tracking of pathogen-recognizing receptors on live cellular membranes

(collaboration with M. G. Parajo group @ ICFO)





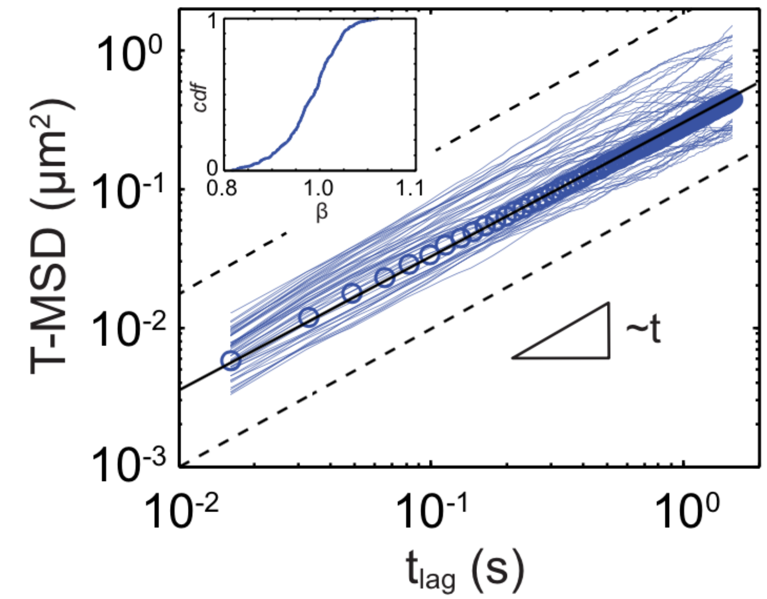
60 frames/s
20nm position accuracy

Sampling of J=600 trajectories

x_j: position of the j-th receptor (1≤j≤J) sampled at N discrete times

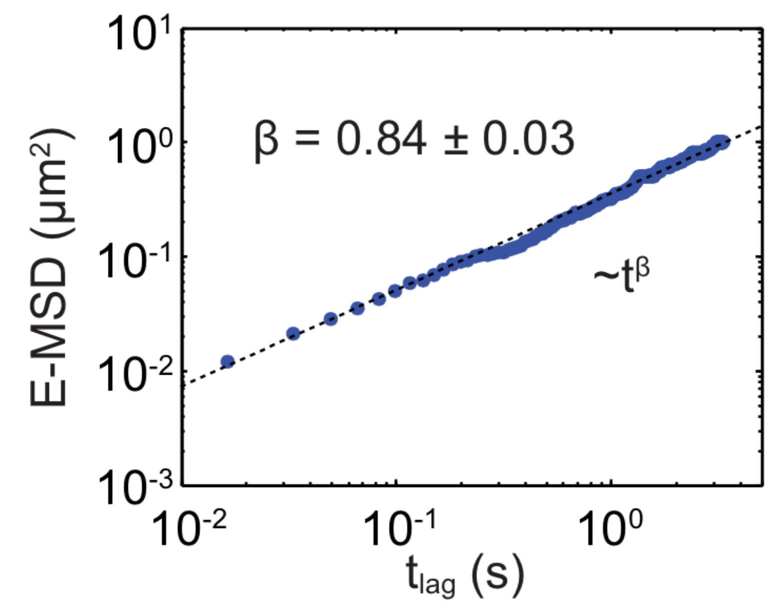
time-averaged mean squared displacement:

$$\text{T-MSD}(t_{lag} = m\Delta t) = \frac{1}{N-m}\sum_{i=1}^{N-m}\left(x_j\left(t_i + m\Delta t\right) - x_j\left(t_i\right)\right)^2$$



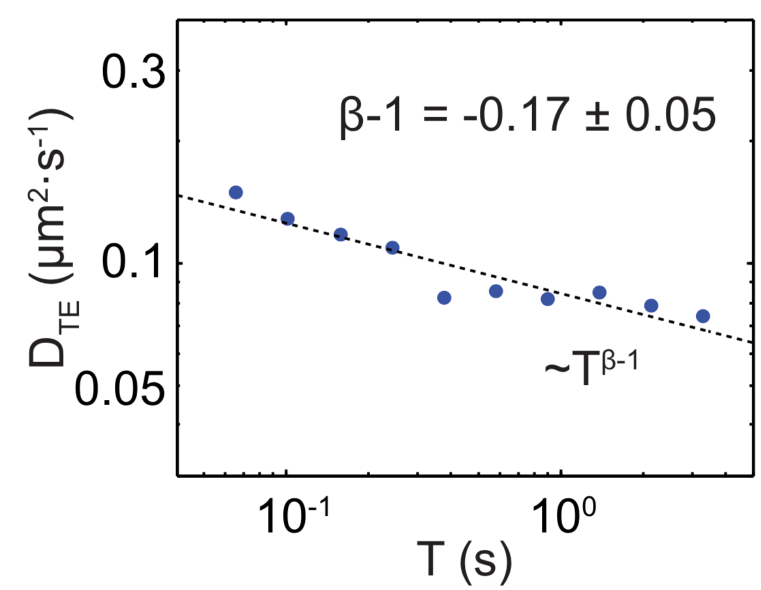
ensemble-averaged mean squared displacement:

$$\text{E-MSD}(t_{lag} = m\Delta t) = \frac{1}{J}\sum_{j=1}^{J}\left(x_j\left(t_i + m\Delta t\right) - x_j\left(t_i\right)\right)^2$$



T-MSD scales differently from E-MSD ➟ weak ergodicity breaking!

$$\text{TE-MSD}(t_{lag}, T) = \frac{1}{J}\frac{1}{\frac{T}{\Delta t} - m}\sum_{i=1}^{\frac{T}{\Delta t}-m}\sum_{j=1}^{J}\left(x_j\left(t_i + m\Delta t\right) - x_j\left(t_i\right)\right)^2$$



The TE-MSD depends on the total observation time T ➟ non-stationary!

Continuous-Time Random Walk

- CTRW: a fat-tailed distribution of waiting times $\sim t^{-1-\beta}$ with $\beta \leq 1$, so that the average waiting time is infinite, induces non-stationary (thus non-ergodic) subdiffusion, with E-MSD$\sim t^{\beta}$, and TE-MSD scaling as $D_{TE}(T)^{*}t_{lag}$, with $D_{TE} \sim T^{\beta-1}$

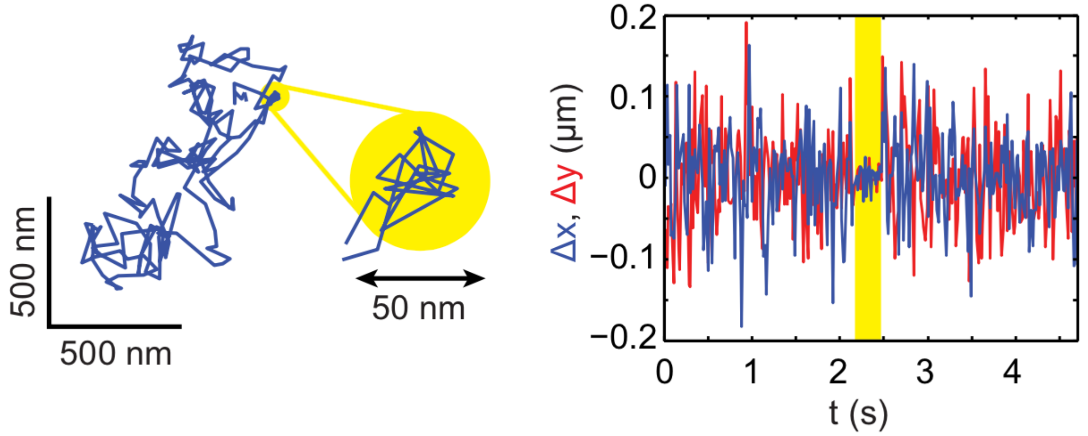
- Widely used model for transport in disordered media (initially developed for amorphous solids)

Montroll & Weiss, 1965; Montroll & Scher, 1973

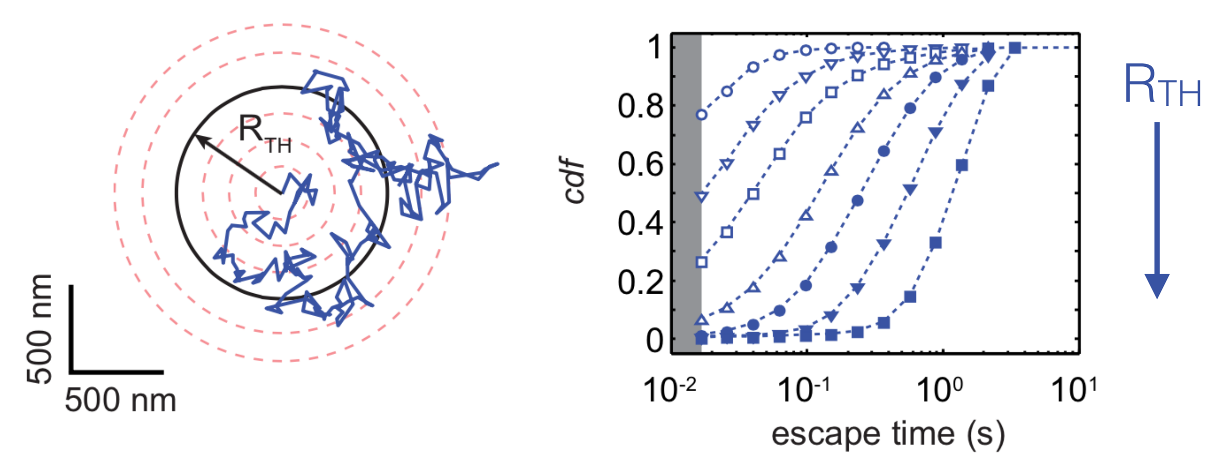
- However, *are trapping events present in the ICFO experiment?*

  only 5% of the trajectories contain events compatible with transient trapping..

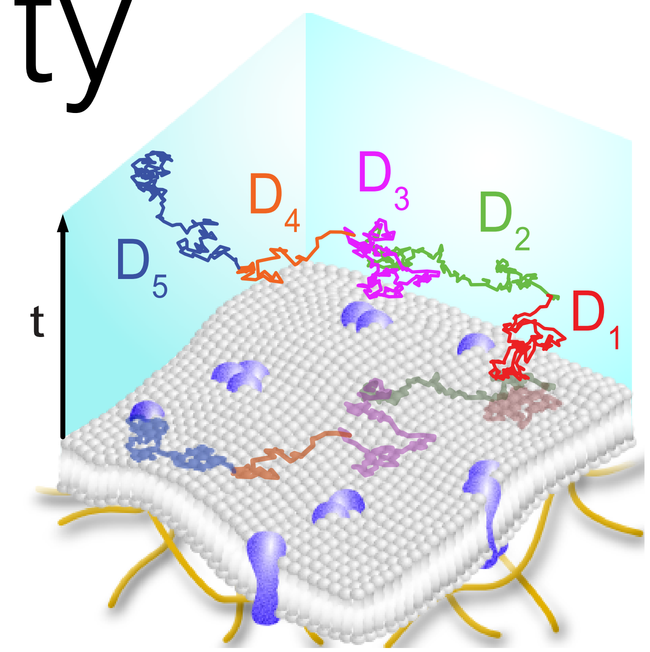
  and excluding these trajectories yields a very similar E-MSD exponent β



- For CTRW, the long-time dynamics is dominated by anomalous trapping events, so that the escape-time distribution should be independent of the trapping radius R_{TH}



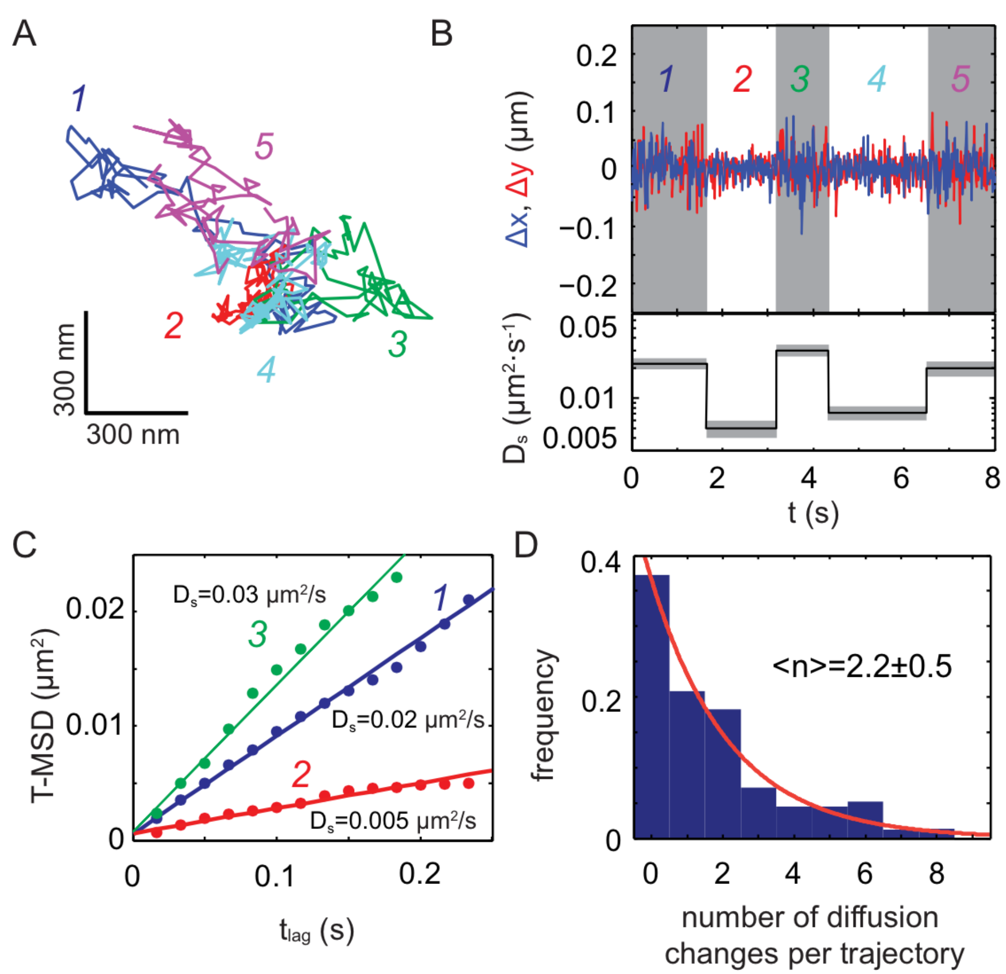
Strongly varying diffusivity



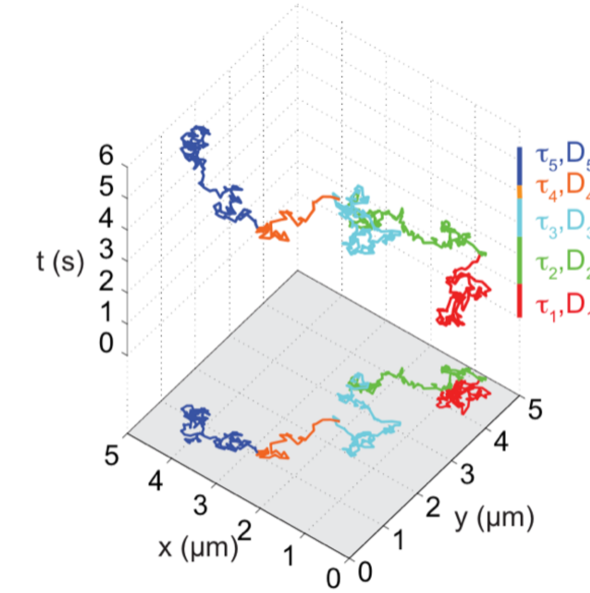
- Maps of receptor motion on the cell membrane highlight the presence of patches with strongly varying diffusivity

- Many possible reasons: crossing regions of low/high viscosity, transient binding, clustering, …

- Employ a likelihood-based Bayesian algorithm to detect time-dependent changes of diffusivity



Theoretical model



- Let's then consider ordinary Brownian motion with a diffusivity that varies randomly, but it's constant on time intervals with random duration (or patches with random sizes)

- Assume a distribution of diffusion coefficients $P_D(D) \sim D^{\sigma-1} e^{-D/b}$

  power law behaviour at small D
  fast decay at large D

  and a conditional distribution of transit times $P_\tau(\tau|D) \sim D^\gamma e^{-\tau D^\gamma / k}$

  mean transit time ~$D^{-\gamma}$
  each area has radius r~$(\tau D)^{-1/2}$

- Three possible regimes:

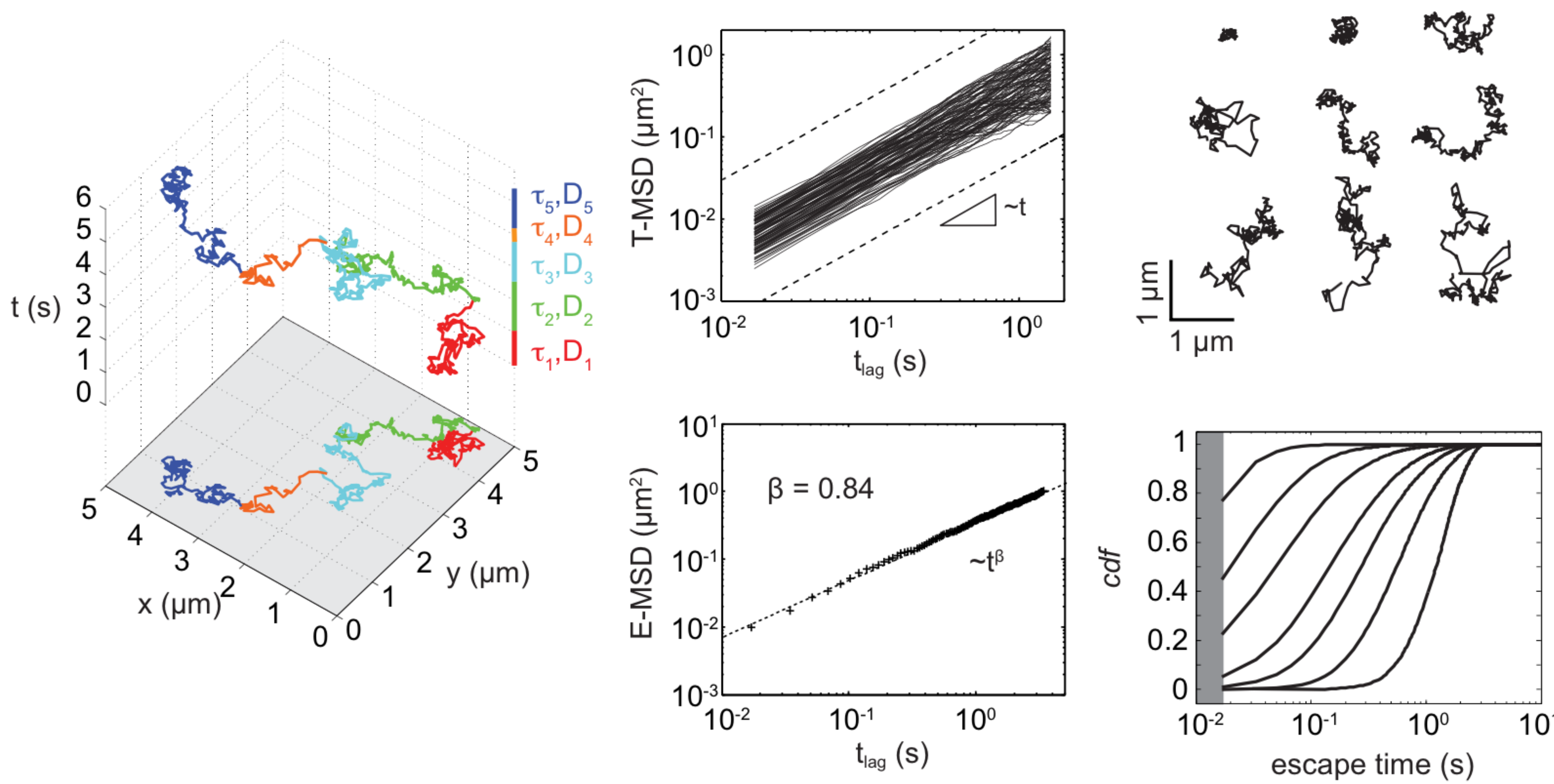
(0) for $\gamma<\sigma$, the long-time dynamics is compatible with regular Brownian motion, and $\beta=1$

(I) for $\sigma<\gamma<\sigma+1$, the average transit-time diverges: non-ergodic sub-diffusion with $\beta=\sigma/\gamma$

(II) for $\gamma>\sigma+1$, both $\langle\tau\rangle$ and $\langle(r_{area})^2\rangle$ diverge: non-ergodic sub-diffusion with $\beta=1-1/\gamma$

PM, Manzo, Torreno-Pina, García-Parajo, Lewenstein & Lapeyre, Phys. Rev. Lett.(2014)

Comparison with experiments



σ=1.16
γ=1.38
regime (I)

Instead, mutated receptors with impaired function (Δ_{rep}) may be best simulated with γ<σ, i.e., they perform usual ergodic and diffusive Brownian motion

Transport properties strongly linked to molecular function

Manzo, Torreno-Pina, PM, Lapeyre, Lewenstein & García-Parajo, Phys. Rev. X (2015)

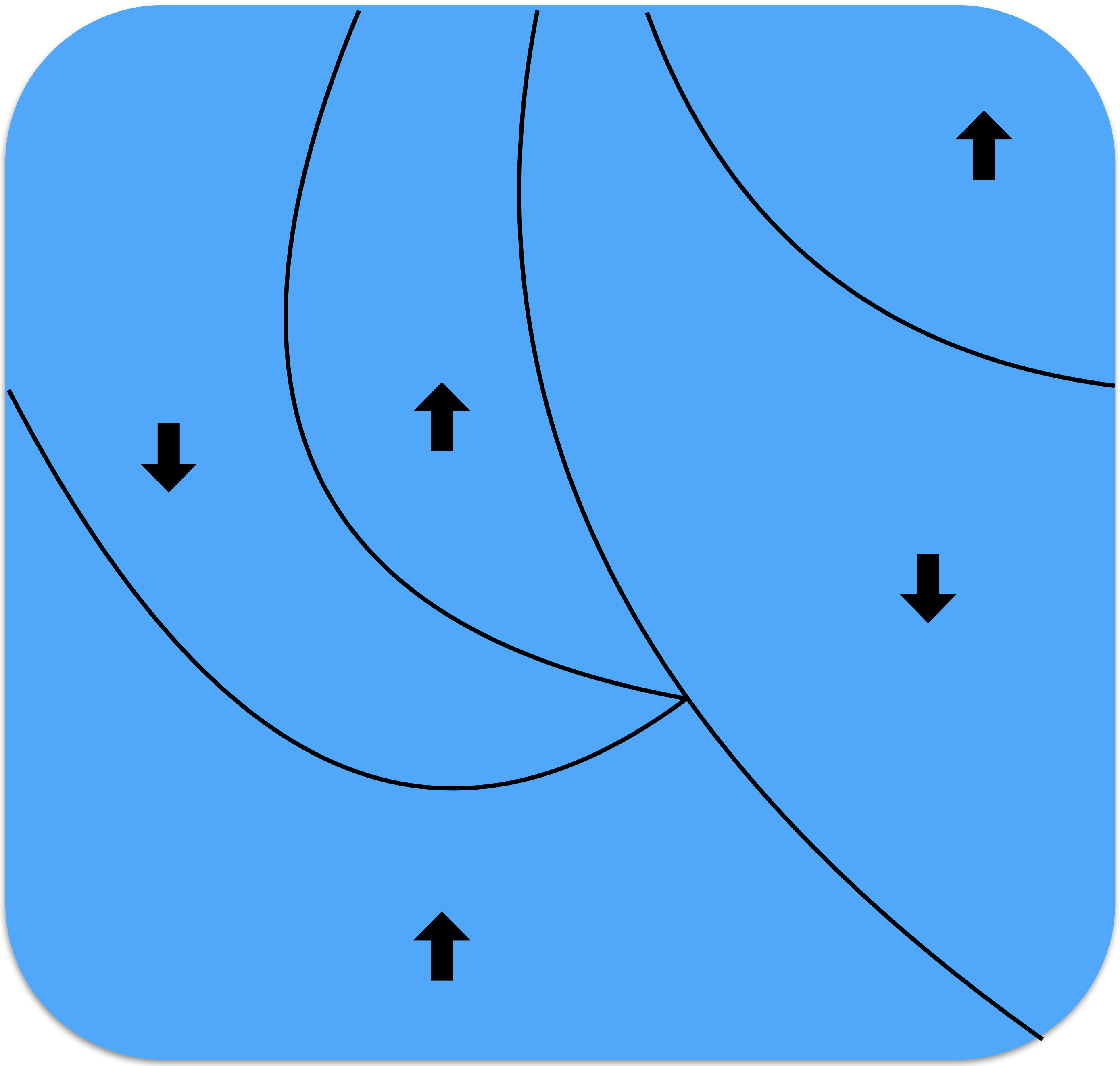
Include receptor/membrane interactions?

Model the membrane as an Ising lattice: $H = -J\sum_i \sigma_i \sigma_{i+1}$ $\sigma_i = \pm 1$

Link diffusivity to local membrane state

(e.g. faster diffusion on an ⬆ background)

Allow walker to locally modify
the membrane state



Kinetic Ising Model for the membrane

- Kinetic Ising Model: $\dot{P}(\sigma,t) = \sum_{\sigma'}[w(\sigma' \to \sigma)P(\sigma',t) - w(\sigma \to \sigma')P(\sigma,t)]$ w: transition rate
- Interacting membrane: $w(\sigma_i \to -\sigma_i) = \frac{\alpha}{2}\left[1 - \frac{\gamma}{2}\sigma_i(\sigma_{i-1}+\sigma_{i+1})\right]$
  α/2: single spin-flip rate
  γ>0: IFM
  γ<0: AFM
- Detailed balance at equilibrium: $P_{\mathrm{eq}}(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_n)w(\sigma_i \to -\sigma_i) = P_{\mathrm{eq}}(\sigma_1,\ldots,-\sigma_i,\ldots,\sigma_n)w(-\sigma_i \to \sigma_i)$

exp(±x)=cosh(x)±sinh(x)

- $P_{\mathrm{eq}}(\sigma_1,\ldots,\sigma_i,\ldots,\sigma_n) \propto \exp[\beta J\sigma_i(\sigma_{i-1}+\sigma_{i+1})] = \cosh[\beta J(\sigma_{i-1}+\sigma_{i+1})]\left\{1 + \frac{\tanh(2\beta J)}{2}\sigma_i(\sigma_{i-1}+\sigma_{i+1})\right\}$
- $P(\sigma,t) = \frac{1}{2^N}\sum_{\{\sigma'\}}(1+\sigma_1\sigma_1')\ldots(1+\sigma_N\sigma_N')P(\sigma',t) = \frac{1}{2^N}\left\{1 + \sum_i \sigma_i q_i(t) + \sum_{i\neq k}\sigma_i\sigma_k r_{i,k}(t) + \ldots\right\}$
  $q_i(t) = \langle\sigma_i(t)\rangle$
  $r_{i,k}(t) = \langle\sigma_i(t)\sigma_k(t)\rangle$
- Recursive system of differential equations: $\alpha^{-1}\dot{q}_i(t) = -q_i(t) + \gamma/2[q_{i-1}(t) + q_{i+1}(t)]$
- Exact time-dependent solutions may be found for: infinite lattice, single spin fixed, (time-delayed) spin correlations, spin systems in a magnetic field, …

R. Glauber, *Time-dependent statistics of the Ising model*, J. Math. Phys. (1963)

Kinetic Ising Model for membrane+walker

- Master equation: $\dot{P}(\sigma, x, t) = [\gamma_s \mathcal{L}_s + \gamma_w \mathcal{L}_w] P(\sigma, x, t)$

s: spins (=membrane)
w: walker
x: position of the walker

- The walker may act as a localized potential for the spins, or it may change the tunneling between the neighboring spins

- If either the spins or the walker dynamics is fast, we may use an adiabatic approximation

- E.g., $\gamma_s \gg \gamma_w \quad \to \quad \mathcal{L} = \gamma_s \left[\mathcal{L}_s + \frac{\gamma_w}{\gamma_s} \mathcal{L}_w \right]$ and $P(\sigma, x, t) \sim P_w(x, t|\sigma) P_s^{(\mathrm{eq})}(\sigma|x)$

    - polaronic behavior (rapid formation of a dressing cloud, dragged along by the walker)

- In the opposite limit instead, the walker spreads very fast, and should act as some kind of diffuse potential (mean-field) for the membrane

(Quantum) Kinetic Ising Models

- Ansatz: $P(\sigma, t) = \sqrt{P_{\mathrm{eq}}(\sigma)}\phi(\sigma, t)$
- The KIM master equation may be written as $\dot{\phi}(\sigma, t) = -\sum_{\sigma'} H_{\sigma\sigma'}\phi(\sigma', t)$

$$H_{\sigma\sigma'} = \sum_{\sigma''} w(\sigma \to \sigma'')\delta_{\sigma\sigma'} - w(\sigma' \to \sigma)\sqrt{\frac{P_{\mathrm{eq}}(\sigma')}{P_{\mathrm{eq}}(\sigma)}}$$

- If the KIM satisfies a detailed-balanced condition, then H is a (real) symmetric matrix, so that the KIM-ME may be seen as a Schrödinger equation in imaginary time, which converges exponentially to the thermal equilibrium solution. Diagonalization of H is then equivalent to the complete solution of the KIM-ME
- Classical spins may be promoted to non-commuting Pauli matrices to obtain quantum models
- Quantum states built from thermal states of classical Hamiltonians, e.g., $|\Psi\rangle = \frac{1}{\sqrt{Z_N}}\sum_{\sigma} e^{-\beta H(\sigma)/2}|\sigma\rangle$ *fulfill the area law in any dimension, even at criticality*, and can be represented efficiently as matrix product states (MPS), or projected entangled pair states (PEPS)

Augusiak, Cucchietti, Haake & Lewenstein, New J. Phys. (2010)

Quantum Brownian Motion

- A small system interacting with a large thermal bath: $H = H_S + H_B + H_I$

- Simplest interaction: $H_I = -\sum_k \kappa_k x_k x$

x_k: position of the k[th] oscillator
κ_k: coupling constant
x: position of the system

- ME for the reduced density matrix: $\dot{\rho}_S(t) = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}s\, \mathrm{Tr}_B[H_\mathrm{I}(t), [H_\mathrm{I}(s), \rho(s)]]$

- Born approx.: $\rho(t) \simeq \rho_S(t) \otimes \rho_B(0)$

- Markov approx.: the bath evolves on timescales much faster than the system, and as such effectively retains no memory of the system's dynamics

- The spectral density contains all details of the bath-system coupling: $J(\omega) \equiv \sum_k \frac{\kappa_k^2}{2m_k\omega_k}\delta(\omega - \omega_k)$

Born-Markov QME

- Weak-coupling limit (second-order perturbation in H_I)

- Ohmic spectral density with Lorentz-Drude cut-off: $J(\omega) = \frac{m\gamma\omega}{\pi} \frac{\Lambda^2}{\omega^2 + \Lambda^2}$

- Consider a particle of mass m in a harmonic trap of frequency Ω. In the high-temperature limit $k_B T \gg \hbar\Lambda \gg \hbar\Omega$ one obtains the celebrated Caldeira-Leggett QME:

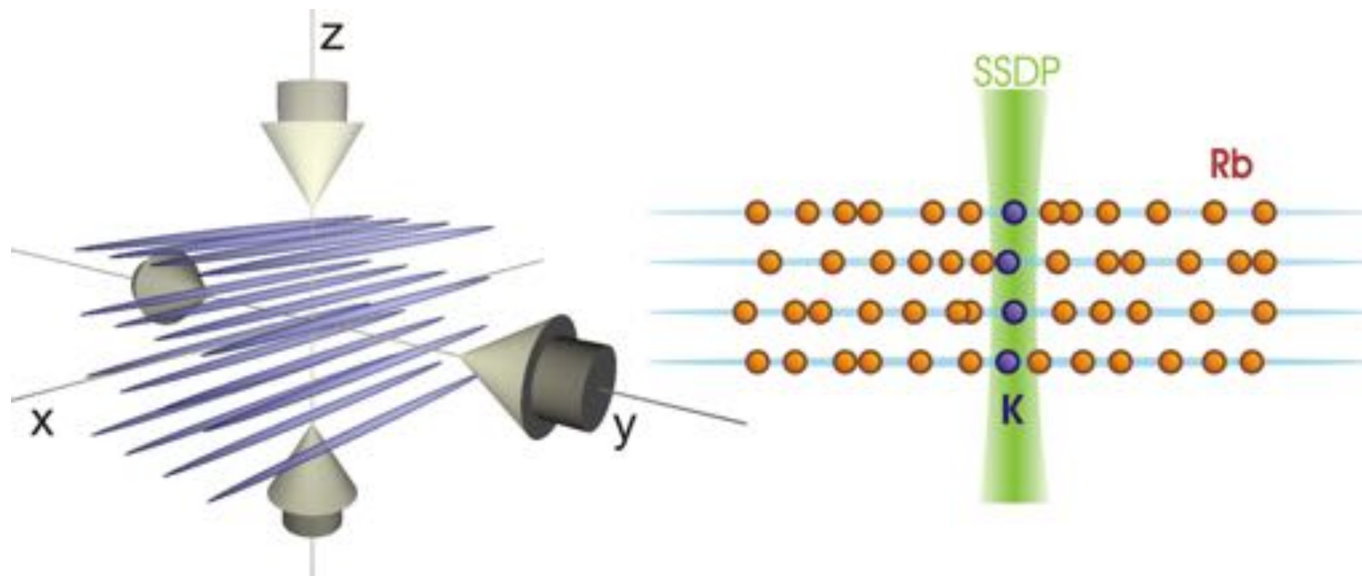
$$\dot{\rho}_S = -\frac{i}{\hbar}[H_{\mathrm{sys}}, \rho_S] - \frac{i\gamma}{2\hbar}[x, \{p, \rho_S\}] - \frac{m\gamma k_B T}{\hbar^2}[x, [x, \rho_S]]$$

momentum damping normal diffusion

- A quantum stochastic process is said to be Markovian only if the underlying noise is white, and if the process is described by a time-independent ME of the Lindblad form

- Note that there exists an *exact* solution of the QBM problem, which however has a time-dependent Liouvillian. Strictly-speaking, QBM is not a quantum Markov process.

Impurities in an inhomogeneous 1D BEC



Catani et al., PRA (2012)

How to treat a spatially-dependent profile of the bath?
Employ a coupling which is a non-linear function of the system's position: $H_I = -\sum_k \kappa_k x_k f(x)$

PM, Lampo, Wehr & Lewenstein, PRA (2015, in press)

How to deal with the low temperatures of a quantum gas?
Non-Markovian effects may become important here!

And what if the spectral density is not Ohmic? (i.e., the noise is not white)

Outlook

A. Classical

- Anomalous diffusion in biological systems

Q: mechanisms leading to sub- or super-diffusion, Levy flights, non-stationarity and non-ergodicity?

- Kinetic Ising Models

Q: mechanisms generating interactions between walkers and substrates?

B. Quantum

- Quantum Kinetic Ising Models

Q: novel exactly or efficiently solvable quantum Hamiltonians?

- Quantum Brownian Motion

Q: models for quantum diffusion in complex or inhomogeneous media?
importance of non-Markovian effects at low temperatures?