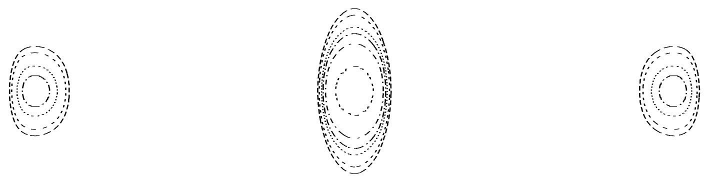
1D model for the dynamics and expansion of elongated Bose-Einstein condensates



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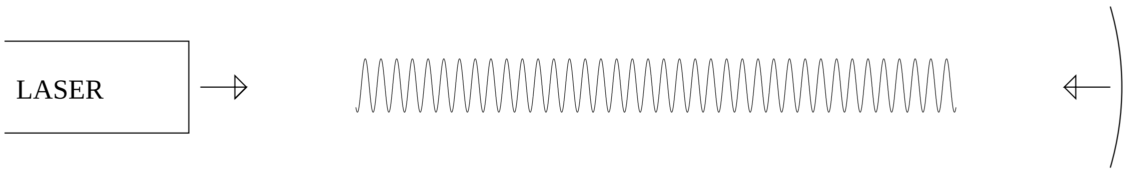
A starting point: BECs in 1D optical lattices

Periodic potentials are powerful tools to investigate coherence properties.

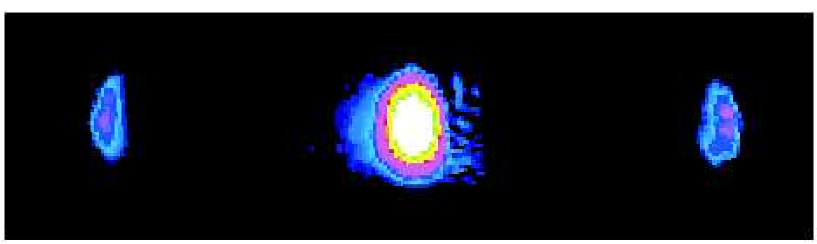


Via the dipole force exerted by an off-resonant laser standing wave it is possible to produce an almost perfect and infinite periodic potential:

$$U_{1D}(r_z) = s \cdot E_r \cdot \cos^2\left(\frac{2\pi r_z}{\lambda_{opt}}\right)$$

$\left(\text{recoil energy: } E_r \equiv h^2/2m\lambda_{opt}^2\right)$

condensates $\oplus$ lattices
- $\nearrow$ pulsed atom laser
- $\longrightarrow$ superfluidity and Josephson effects
- $\searrow$ matter diffraction



¿Qubits arrays?

Why a 1D model?

* BECs at $T = 0$ are usually described within the framework given by the Gross-Pitaevskii Equation (1961):

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r},t) + gN\left|\Psi(\mathbf{r},t)\right|^2\right]\Psi(\mathbf{r},t)$$

$$g \equiv \frac{4\pi\hbar^2 a}{m} \qquad (a: \text{ scattering length})$$

* The absence of analytical solutions often implies a numerical approach

but

rapid spatial potential variations $\Longrightarrow$ heavy numerical simulations

$\Downarrow$

uni-dimensional geometry:
cigar-shaped condensates $\longrightarrow$ **need for a 1D model** ✍
and axial dynamics

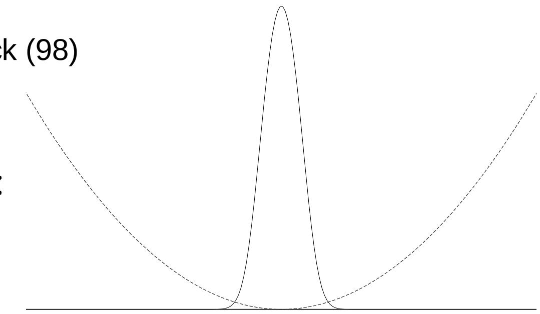
$$1D: \quad i\hbar\frac{\partial}{\partial t}\psi(z,t) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + U(z,t) + \textbf{?}\right\}\psi(z,t)$$

3 recent proposals for the dimensionality reduction

Statical renormalization methods

I Weak auto-interaction Jackson, Kavoulakis and Pethick (98)

$$\frac{Na}{a_{\bar{\omega}}} \ll 1 :$$



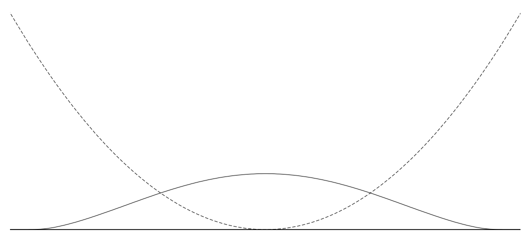
$$i\hbar \frac{\partial}{\partial t}\psi(z,t) = \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + U(z) + \frac{gN}{\pi < R^2 >} |\psi(z,t)|^2 \right\} \psi(z,t)$$

$$\left(a_\perp \equiv \sqrt{\frac{\hbar}{m\omega_\perp}}, \quad < R^2 > = < X^2 > + < Y^2 > = 2a_\perp^2 \right)$$

II Strong auto-interaction Trippenbach, Band and Julienne (00)

(Thomas-Fermi limit)

$$\frac{Na}{a_{\bar{\omega}}} \gg 1 :$$



$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left\{ \sum_{i=1}^{d} \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{m}{2}\omega_i^2 x_i^2 \right) + G_0^{(d)} |\Psi(\mathbf{r},t)|^2 \right\} \Psi(\mathbf{r},t)$$

with

$$G_0^{(d)} = G_0^{(d)}(N, a, \bar{\omega})$$

Dynamical renormalization method

III Factorized wave-function, gaussian transverse part[a]:

Ansatz: $$\begin{cases} \Psi(\mathbf{r},t) = \phi\Big(r,t;\sigma_o(z,t)\Big)\psi(z,t) \\ \phi\Big(r,t;\sigma_o(z,t)\Big) = \dfrac{1}{\sqrt{\pi}\sigma_o} e^{-r^2/2\sigma_o^2} \end{cases}$$

$$S[\Psi] = \int \mathrm{d}t \int \mathrm{d}z\, \mathrm{d}^2\mathbf{r}\; \Psi^* \left\{ i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - \frac{1}{2}m\omega_\perp^2 r^2 - U(z,t) - \frac{gN}{2}|\Psi|^2 \right\} \Psi$$

Inserting the Ansatz in the action of the system, under the **"slowly varying approximation"** $\nabla^2\phi \approx \nabla_\perp^2\phi$ one gets a 1D dynamical equation for the axial wave-function (NPSE):

$$i\hbar\frac{\partial}{\partial t}\psi(z,t) = \left\{ -\frac{\hbar^2}{2m}\nabla_z^2 + U(z,t) + \frac{gN}{2\pi\sigma_o^2}|\psi(z,t)|^2 + \right.$$
$$\left. + \frac{\hbar\omega_\perp}{2}\left(\frac{a_\perp^2}{\sigma_o^2} + \frac{\sigma_o^2}{a_\perp^2}\right) \right\} \psi(z,t)$$

where the transverse width is given by: $\sigma_o(z,t) = a_\perp \sqrt[4]{1 + 2aN|\psi(z,t)|^2}$

⌘ The NPSE (III) gives an axial description of a condensate in a time-independent harmonic trap much more accurate than (I) and (II).

[a] L. Salasnich, Laser Phys. **12**, 198 (2002);
. L. Salasnich, A. Parola and L. Reatto, Phys. Rev. A **65**, 043614 (2002).

Free expansion and collective modes: need for a new equation

- (I), (II) and (III) could be used to describe the ground state but **not** the free expansion of a condensate:
  - ✗ (I), (II): for a *cigar-shaped* condensate ($\omega_\perp \gg \omega_z$) in the TF limit a generic (statically renormalized) 1D-GPE **largely** overestimates the axial width of the freely expanding wave-packet (see Fig. 5, coming soon);
  - ✗ (III) is derived in presence of a constant non-zero transverse harmonic confinement:

$$U_\perp^{ho} \to 0 \;\Rightarrow\; a_\perp \to \infty \;\Rightarrow\; \sigma_o \to \infty .$$

- The interplay between the axial and radial dynamics is necessary to account for the quadrupole oscillations.

✔ Solution

$$\left.\begin{array}{c}\left\langle \begin{array}{c}\text{coordinates rescaling}\\ \text{gauge transform}\end{array}\right\rangle \\ \text{gaussian transverse hypothesis}\end{array}\right\} \rightsquigarrow \textit{dr}\text{-GPE}$$

Scaling Ansatz

Introduction of rescaled coordinates: $\mathbf{x} \equiv \frac{\mathbf{r}}{\boldsymbol{\lambda}}$

Local gauge transformation: $\Psi(\mathbf{r},t) = e^{\frac{im}{2\hbar}\sum r_j^2 \frac{\dot{\lambda}_j}{\lambda_j}} \frac{\tilde{\Psi}(\mathbf{x},t)}{\sqrt{\lambda_x(t)\lambda_y(t)\lambda_z(t)}}$

$$\begin{cases} \frac{\ddot{\lambda}_j(t)}{\lambda_j(t)} + \omega_j^2(t) \equiv \frac{\omega_j^2(0)}{\lambda_j^2(t)\lambda_x(t)\lambda_y(t)\lambda_z(t)} \\ \boldsymbol{\lambda}(0) = 1,\ \dot{\boldsymbol{\lambda}}(0) = 0 \implies \tilde{\Psi}(\mathbf{x},0) \equiv \Psi(\mathbf{r},0) \end{cases}$$

GPE $\rightarrow$ $i\hbar\partial_t \tilde{\Psi}(\mathbf{x},t) = \left\{ \dots + \frac{m}{2}\sum_j \lambda_j^2 x_j^2 \left(\frac{\ddot{\lambda}_j(t)}{\lambda_j(t)} + \omega_j^2(t) \right) + \dots \right\} \tilde{\Psi}(\mathbf{x},t)$

$$i\hbar\partial_t \tilde{\Psi}(\mathbf{x},t) = \left\{ -\frac{\hbar^2}{2m}\sum_j \frac{\partial_{x_j}^2}{\lambda_j^2(t)} + \frac{U_{ho}(\mathbf{x},0) + gN|\tilde{\Psi}(\mathbf{x},t)^2|}{\lambda_x(t)\lambda_y(t)\lambda_z(t)} \right\} \tilde{\Psi}(\mathbf{x},t)$$

$\hookrightarrow$ | total elimination of the harmonic potential temporal dependency |

☞ Since the performed transformation is unitary, the latter equation is exact.

$U_{ho}(t) \rightsquigarrow U_{ho}(0)$ ↬ ⟨ the rescaled wave-function $\tilde{\Psi}$ evolves in a fictitious harmonic potential, whose characteristic lenghts are fixed to their $t = 0$ values ⟩

Y. Castin e R. Dum, Phys. Rev. Lett. **77**, 5315 (1996);

. Yu. Kagan, E. L. Surkov and G. V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996).

Deduction of the *dr*-GPE

Assuming a cylindrically-symmetric potential, sum of a time-dependent harmonic term and an additional axial component:

$$U(\mathbf{r},t) = \frac{m}{2} \sum_{j=z,\perp} \omega_j^2(t) r_j^2 + U_{1D}(r_z,t)$$

we impose the gaussian factorization on the rescaled wave-function:

$$\textit{Ansatz}: \begin{cases} \tilde{\Psi}(\mathbf{x},t) = \tilde{\phi}\Big(x,y,t;\sigma(z,t)\Big)\tilde{\psi}(z,t) \\ \tilde{\phi}\Big(x,y,t;\sigma(z,t)\Big) = \frac{1}{\sqrt{\pi}\sigma}\, e^{-\frac{x^2+y^2}{2\sigma^2}} \end{cases}$$

Under the hypothesis $\nabla^2\phi \approx \nabla_\perp^2\phi$, by variational deduction one obtains the dynamical equation for the axial wave-function (***dr*-GPE**):

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}(z) = \Bigg\{ -\frac{\hbar^2}{2m}\frac{\nabla_z^2}{\lambda_z^2} + \frac{\hbar^2}{2m}\frac{1}{\lambda_\perp^2\sigma^2} + U_{1D}(\lambda_z z,t) + \\ + \frac{1}{\lambda_z\lambda_\perp^2}\left[\frac{m\omega_z^2(0)}{2}z^2 + \frac{m\omega_\perp^2(0)}{2}\sigma^2 + \frac{gN}{2\pi\sigma^2}|\tilde{\psi}|^2\right]\Bigg\}\tilde{\psi}(z)$$

where the transverse width is given by: $\sigma(z,t) = a_\perp^o \sqrt[4]{\lambda_z(t) + 2aN|\tilde{\psi}(z,t)|^2}$

$$\left(a_\perp^o \equiv \sqrt{\hbar/m\omega_\perp(0)}\right)$$

The transverse width of the true wave function $|\phi|^2 \propto e^{-r^2/\Sigma^2}$ is given by $\Sigma(r_z,t) \equiv \sigma\cdot\lambda_\perp$

Properties of the *dr*-GPE

d*inamically* ***r****escaled*-**GPE** : $\begin{cases} \text{1D non-linear Schrodinger eq. } (\leftrightsquigarrow \text{ GPE}) \\ \text{function of the rescaled coordinate } z(t) \\ \text{with constant harmonic potential} \end{cases}$

- ☞ The *dr*-GPE is energy conserving and reduces to the NPSE in case of a time-independent harmonic potential.

- ☞ The variational parameter σ allows the model for an intrinsic description of the radial dynamics of the system.

- ☞ The evolution of $\tilde{\Psi}$ due to the harmonic confinement variations is mostly absorbed by the $\langle \ldots \rangle$ transformations (in the TF limit $|\tilde{\Psi}(t)| = |\tilde{\Psi}(0)|$).

- ☞ The fictitious constant harmonic potential gives sense to a gaussian factorization even in the case of a sudden release of the external confinement.

- ☞ Since the numerical solution of the λ equations is straightforward, the propagation of the *dr*-GPE requires the same computational effort of a simple 1D-GPE.

Applications

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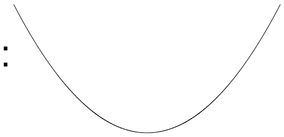
Lattice off: ground state

V(z,0):



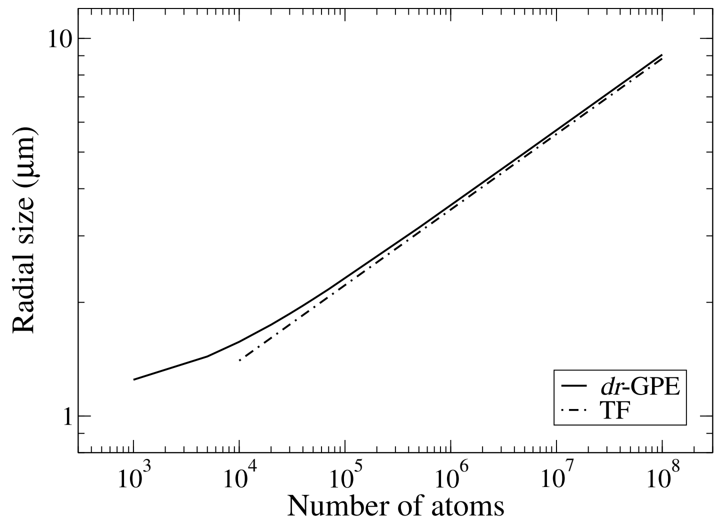


Figure 1: Radial size of the ground state in an harmonic potential as a function of N. The shift between the two curves is expected, since the TF approximation systematically underestimates the condensate radii (especially at low N).

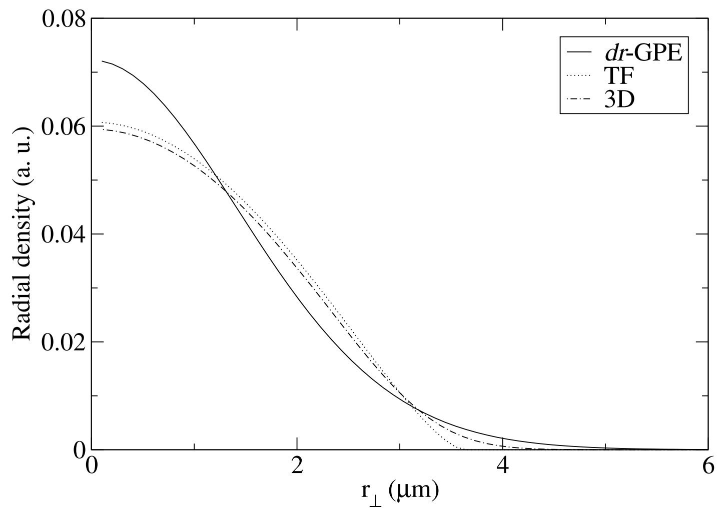


Figure 2: Radial density integrated over the axial coordinate of a condensate in the ground state of an harmonic potential.

We use typical LENS parameters: $2 \cdot 10^4 < N < 2 \cdot 10^5$ ^{87}Rb atoms in cigar-shaped configurations ($\nu_z = 9$ Hz, $\nu_\perp = 92$ Hz) exposed to lattices with $\lambda_{opt} = 795$ nm and $0 \leq s \leq 6$.

An interesting feature of this model is the possibility to describe oscillations induced by modulations of the axial or the radial part of the trapping potential.

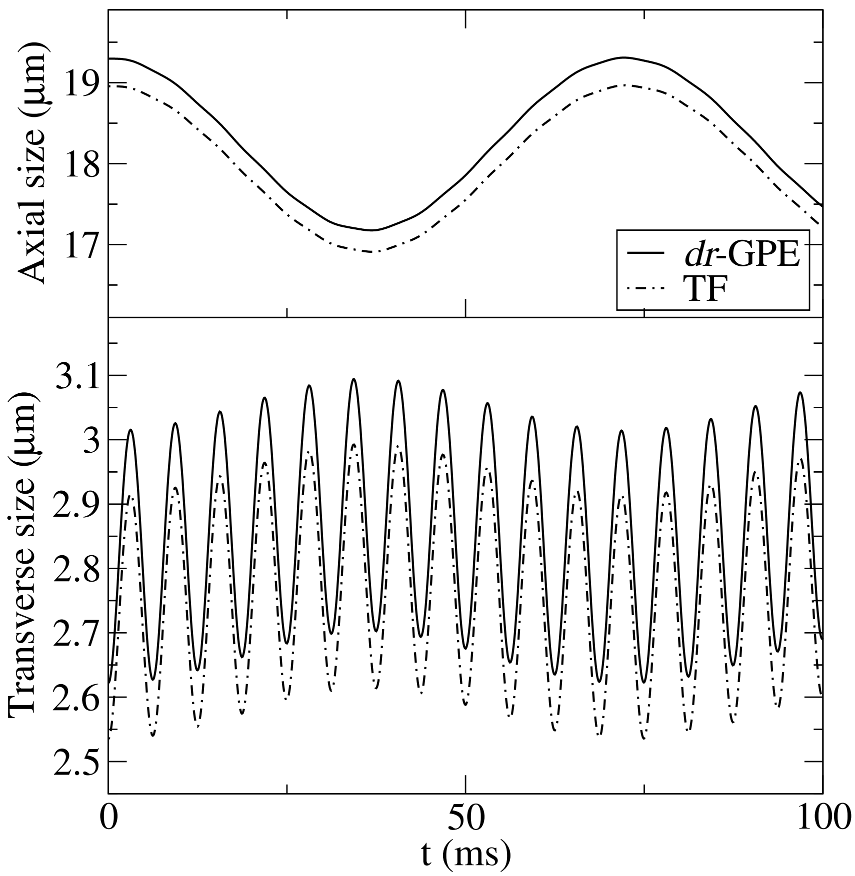


Figure 3: Evolution of the axial (top) and radial (bottom) sizes of a condensate performing shape oscillations, obtained suddenly weakening the radial confi nement:

.
$$\nu_\perp : 92Hz \;\rightarrow\; 80Hz.$$

In both graphs it is possible to observe the superposition of the quadrupole and the faster transverse breathing frequencies ($\omega^Q = \sqrt{5/2}\omega_z,\ \omega^{TB} = 2\omega_\perp$).

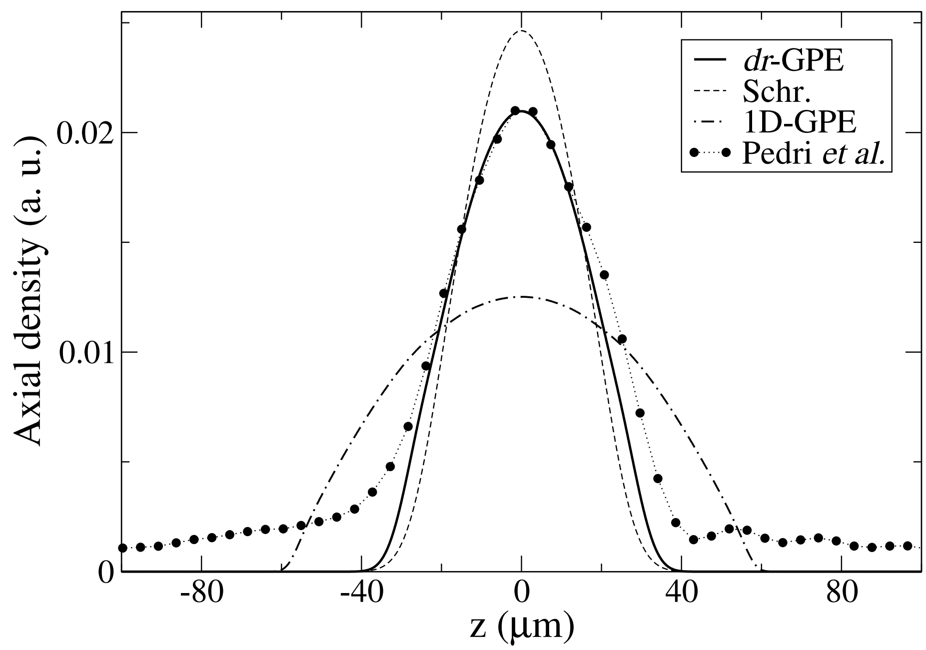


Figure 4: Schrodinger, *dr*-GPE, 1D-TF predictions and experimental values for the axial density of the condensate after a free expansion of $t_{exp} = 29.5$ ms.

TF *dr*-GPE: $i\hbar\partial_t\tilde{\psi}(z) = \left\{ \ldots + \frac{1}{\lambda_z\lambda_\perp^2}\left[\frac{3}{2}\hbar\omega_\perp(0)\sqrt{2aN|\tilde{\psi}|}\right]\right\}\tilde{\psi}(z)$

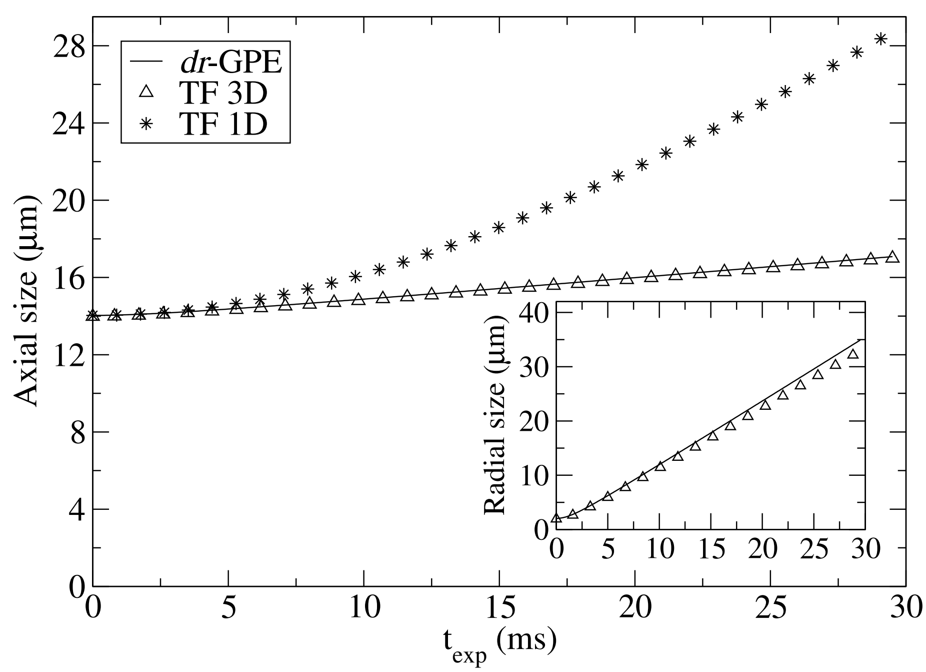
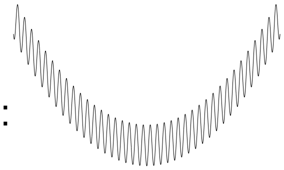


Figure 5: *dr*-GPE, 1D-TF and 3D-TF predictions for the axial and radial sizes of the condensate during a free expansion from an harmonic potential.

Lattice on: ground state

V(z,0): 

Even when a strong lattice causes deep modulations of $|\Psi|^2$, the $\nabla^2\phi \approx \nabla_\perp^2\phi$ approximation is still a good one: the *dr*-GPE results are really close to those of the complete equation.

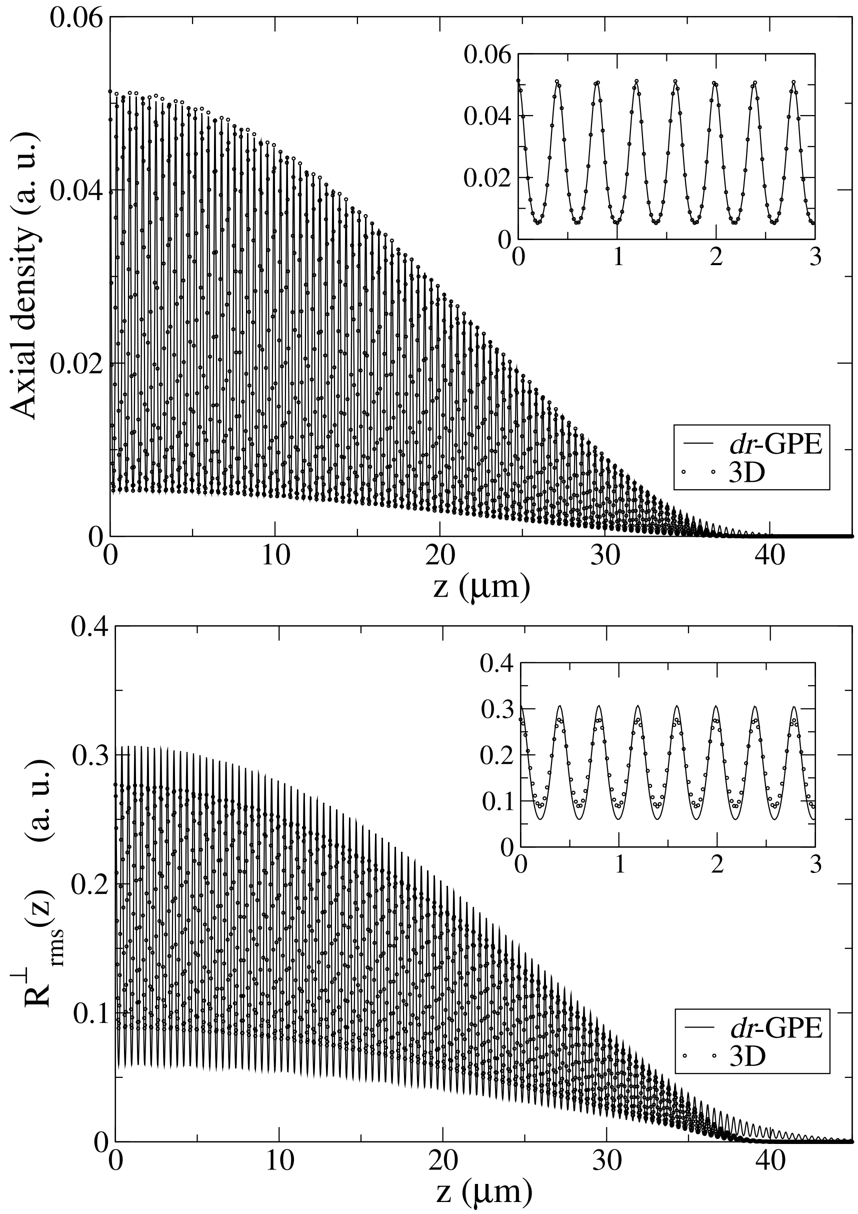
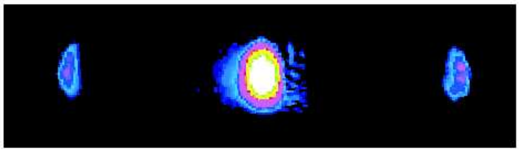


Figure 6: Axial density $\rho(z)$ (top) and rms transverse radius, averaged only on the transverse coordinates and plotted as a function of the axial coordinate (bottom); the insets show a magnifi cation of the central region (lattice intensity $s = 5$).

Free expansion from a combined (harmonic+optical) potential [a]



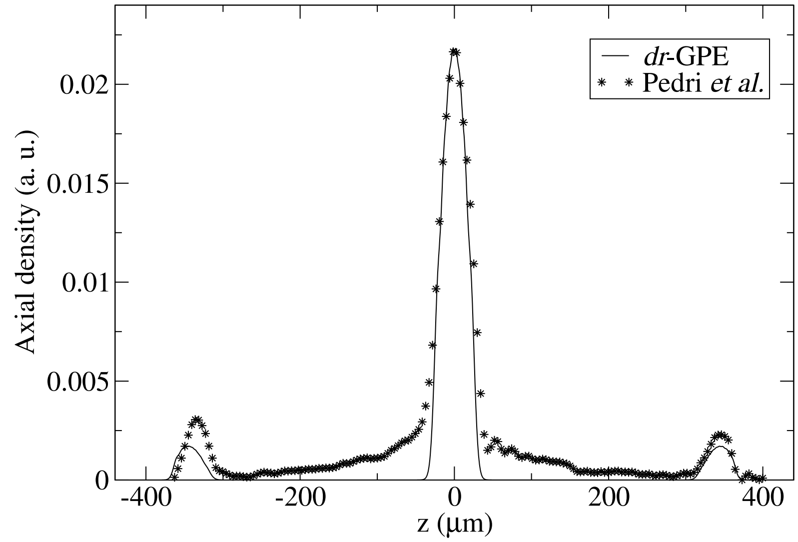


Figure 7: Axial density of a coherent array of condensates, initially trapped in an $s = 5$ optical lattice, imaged after a free expansion of 29.5 ms.

Lattice on: collective modes [b]

The presence of an optical lattice renormalizes the normal mode frequencies[c]:

$$\omega_z^D \rightarrow \sqrt{m/m^*}\omega_z^D$$

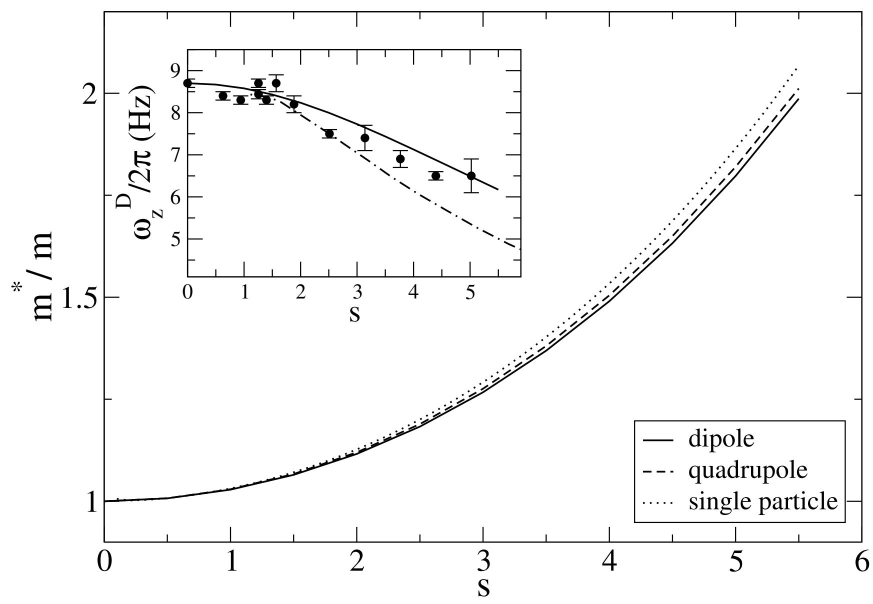


Figure 8: Effective mass m^* as a function of the lattice intensity compared to that of a single particle in a periodic potential. Inset: the frequency of the dipole mode is compared with a recent LENS experiment (the dotted line is their theoretical curve).

[a] P. Pedri, L. Pitaevskii, S. Stringari, C. Fort, F. S. Cataliotti *et al.*, PRL **87**, 220401 (2001).
[b] F. S. Cataliotti, C. Fort, A. Trombettoni, A. Smerzi, M. Inguscio *et al.*, Science **293**, 843 (2001).
[c] M. Kramer, L. Pitaevskii and S. Stringari, Phys. Rev. Lett. **88**, 180404 (2002).

Negative mass effects

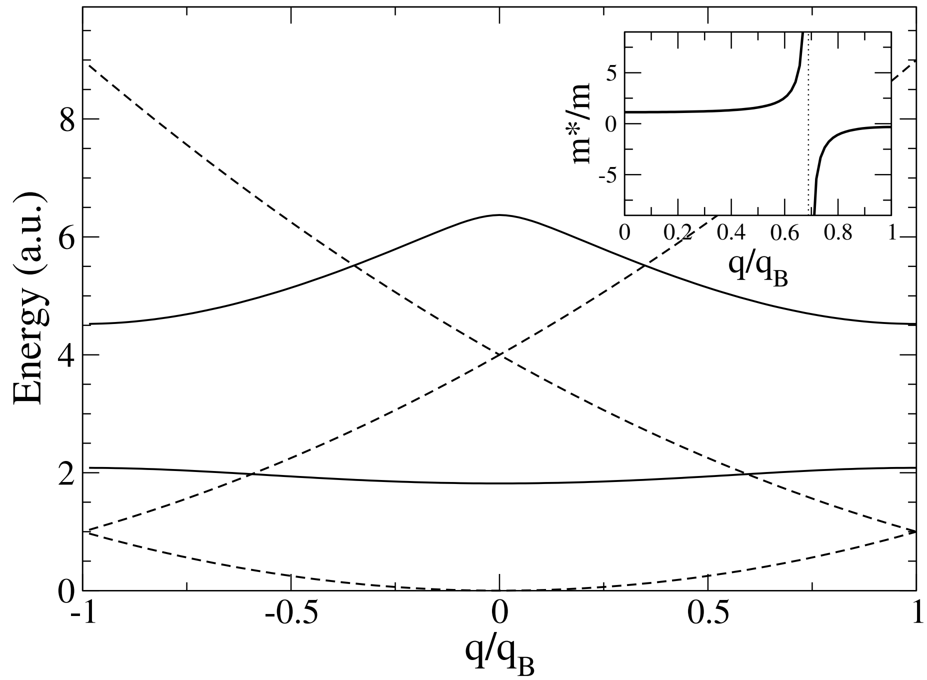


Figure 9: Structure of the lowest two bands for a single particle in a 1D periodic potential with $s = 2$ (solid), compared to the free particle case (dashed). Inset: effective mass (inversely proportional to the band curvature) in the fi rst band.

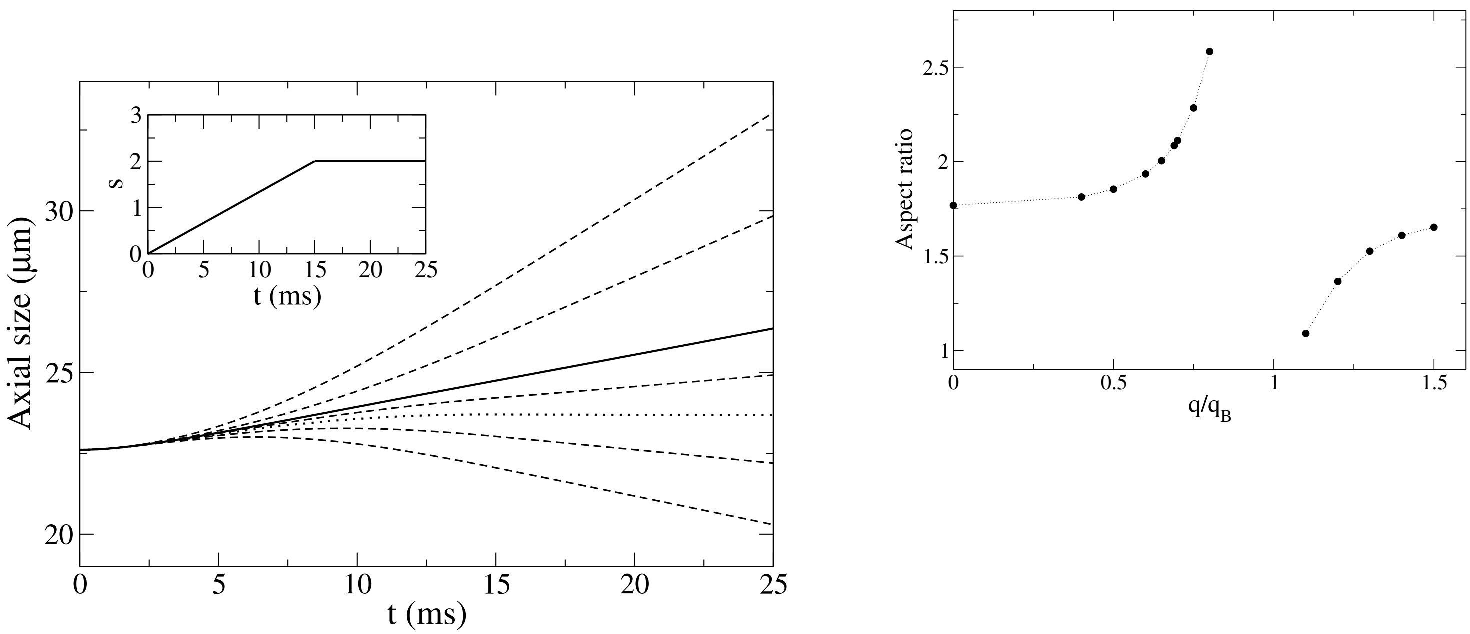


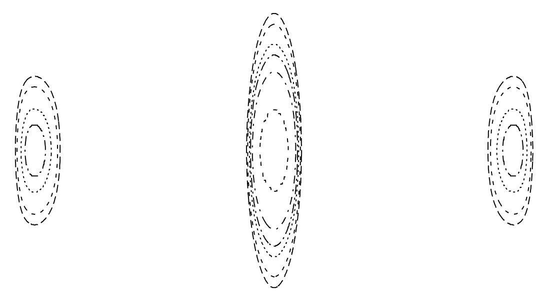
Figure 10: Left: axial width of a condensate adiabatically loaded in a 1D lattice ($s = 2$) for different initial velocities (solid: $q = 0$, dotted: $q = 0.685$), plotted as a function of time. Inset: lattice intensity ramp of the thought experiment.

Right: aspect ratio after 25 ms of expansion, plotted as a function of the initial BEC velocity.

Conclusions

$$\left.\begin{array}{l}\left\langle \begin{array}{c}\text{coordinates rescaling}\\ \text{gauge transform}\end{array}\right\rangle \Rightarrow a_\perp = \text{cost}\\ \text{gaussian transverse hypothesis}\end{array}\right\} \rightsquigarrow \textit{dr}\text{-GPE}$$

***dr*-GPE**: 1D effective equation able to describe ground state and dynamics of BECs confined in *cigar-shaped* harmonic traps, generically time-dependent and containing an arbitrary axial component.

dr-GPE(z) ⇝ 

Applications:

- collective oscillations and excitations
- exploration of new dynamical effects, band spectroscopy
- behaviour in generic 1D potentials (gravity, lattices, barriers)