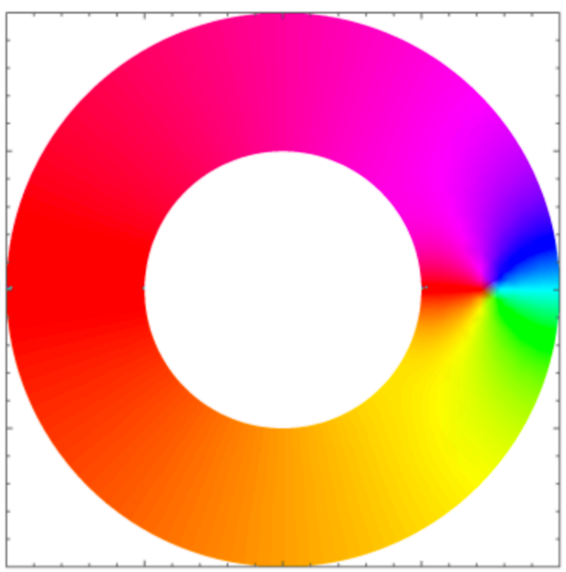
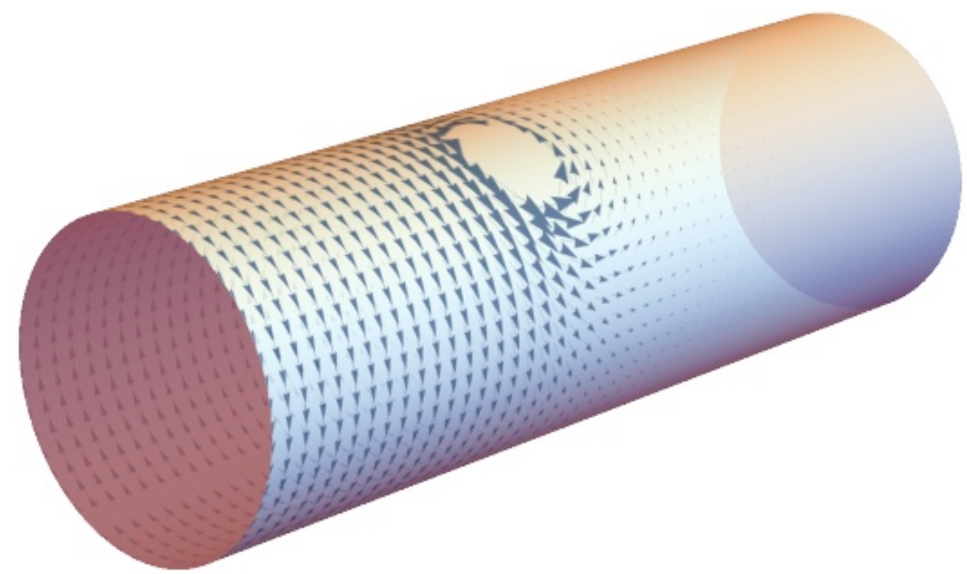
Superfluid vortex dynamics on peculiar surfaces

Pietro Massignan





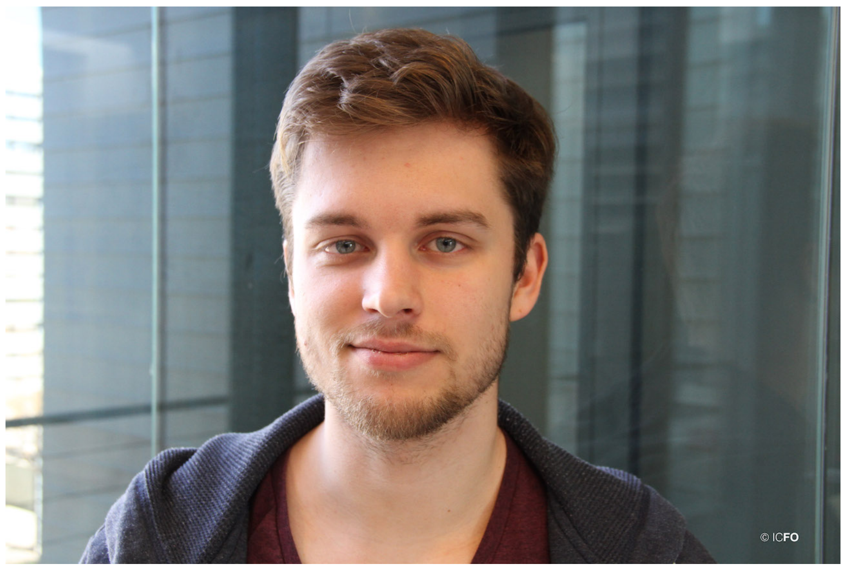
UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

ICFO Institut de Ciències Fotòniques

Outline

- Classical vs. superfluid turbulence
- Potential flow for perfect fluids in 2D
- Vortices on an annulus
- Conformal mappings
- Vortices on a cylinder

Collaborators



Nils Guenther

ICFO
Institut
de Ciències
Fotòniques

Alexander Fetter

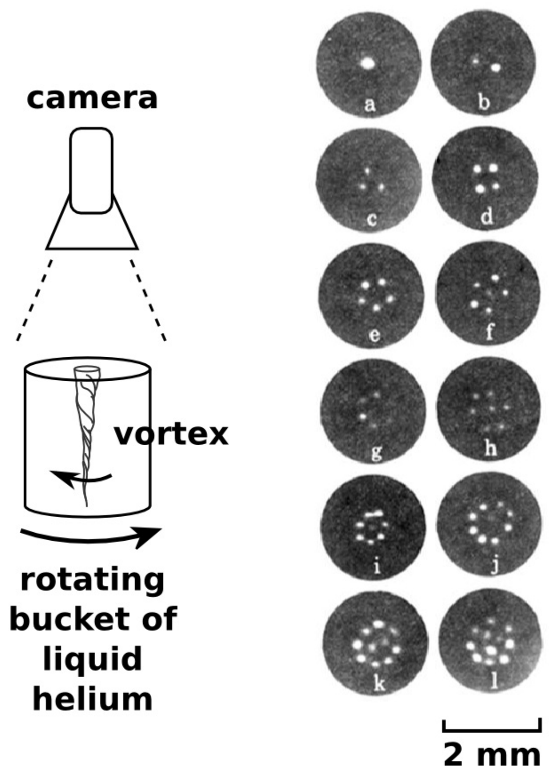
LELAND STANFORD JUNIOR UNIVERSITY
DIE LUFT DER FREIHEIT WEHT
1891

N. Guenther, P. Massignan, and A. Fetter
Phys. Rev. A, in press
arXiv:1708.08903

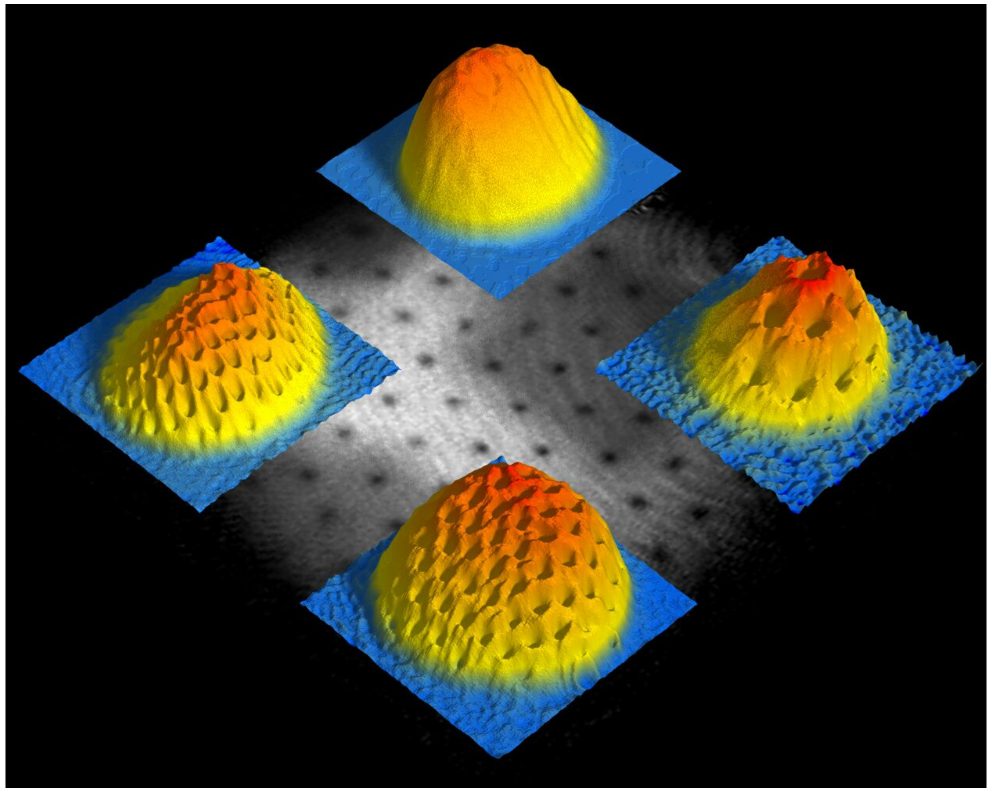
Classical turbulence

chaotic
multiscale
viscous

Superfluid vortices



[Yarmchuk, Gordon and Packard, 1979]



[Ketterle's group @ MIT, 2001]

regular
crystalline
non-viscous

Superfluid hydrodynamics

- Macroscopic condensate wavefunction: $\Psi = \sqrt{n} e^{i\Phi}$
- Superfluid velocity: $\mathbf{v} = \frac{\hbar}{M} \nabla \Phi$
- Vorticity: $\nabla \times \mathbf{v} = \frac{\hbar}{M} \nabla \times \nabla \Phi = 0$ (*irrotational*)
- Quantized circulation: $\oint \mathrm{d}\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{M} \oint \mathrm{d}\mathbf{l} \cdot \nabla \Phi = 2\pi j \frac{\hbar}{M}, \qquad j \in \mathbb{Z}$
- Current conservation: $\nabla \cdot (n\mathbf{v}) = 0$
- For constant density, the fluid becomes *incompressible*: $\nabla \cdot \mathbf{v} = 0$
- SF (non-viscous) + irrotational + incompressible = **perfect fluid**

[A. Fetter, Rev. Mod. Phys. **81**, 647 (2009)]

2D potential flow

- For 2D incompressible fluids, $\mathbf{v} = \left(\frac{\hbar}{M}\right) \hat{\mathbf{n}} \times \nabla\chi$ ← stream function

  and the velocity is parallel to iso-contours of χ
  and orthogonal to iso-contours of Φ

- Perfect fluids in 2D fully described by $F = \chi + i\Phi$

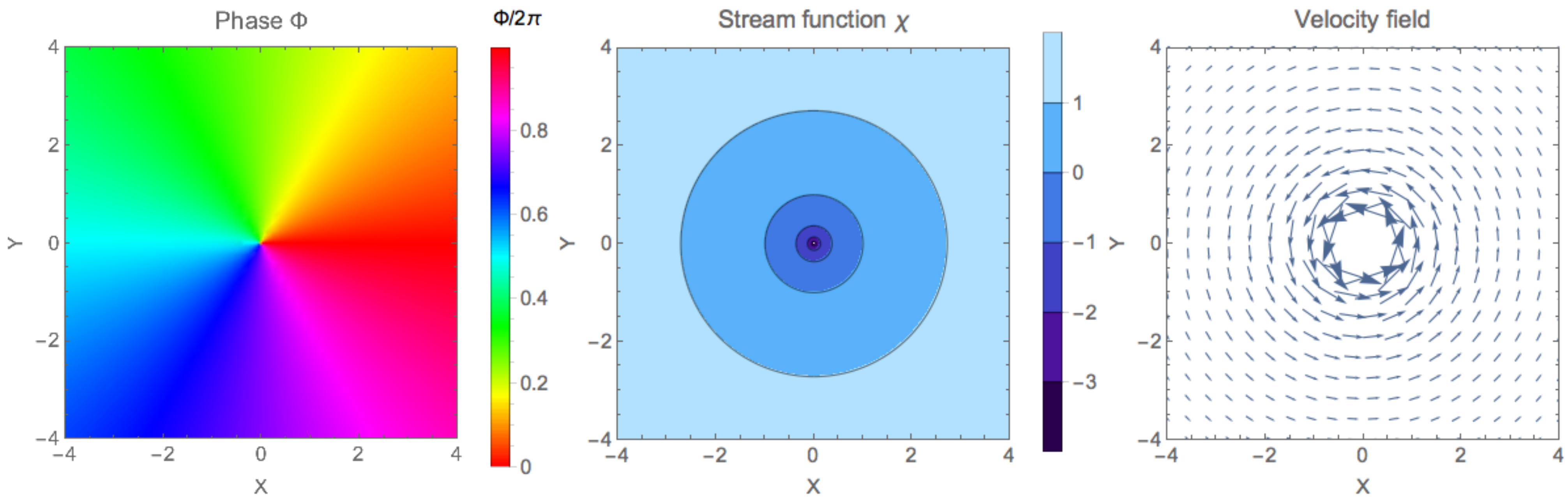
- F is a meromorphic function of $Z = X + iY$

- Cauchy-Riemann conditions readily imply: $v_Y + iv_X = \frac{\hbar}{M}\frac{\partial F}{\partial Z}$

$$\frac{\partial\chi}{\partial x} = \frac{\partial\Phi}{\partial y}\,, \quad \frac{\partial\chi}{\partial y} = -\frac{\partial\Phi}{\partial x}$$

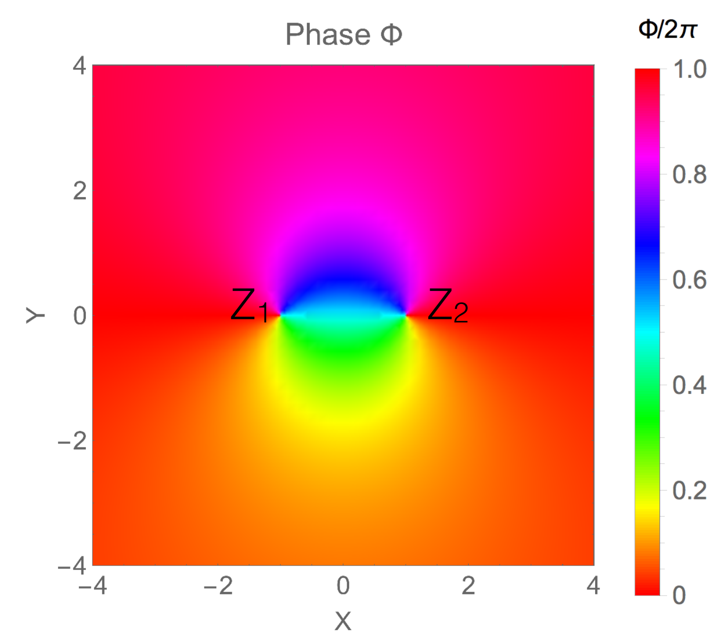
Vortices on a plane

- A single vortex at the origin: $F(Z) = \log(Z)$



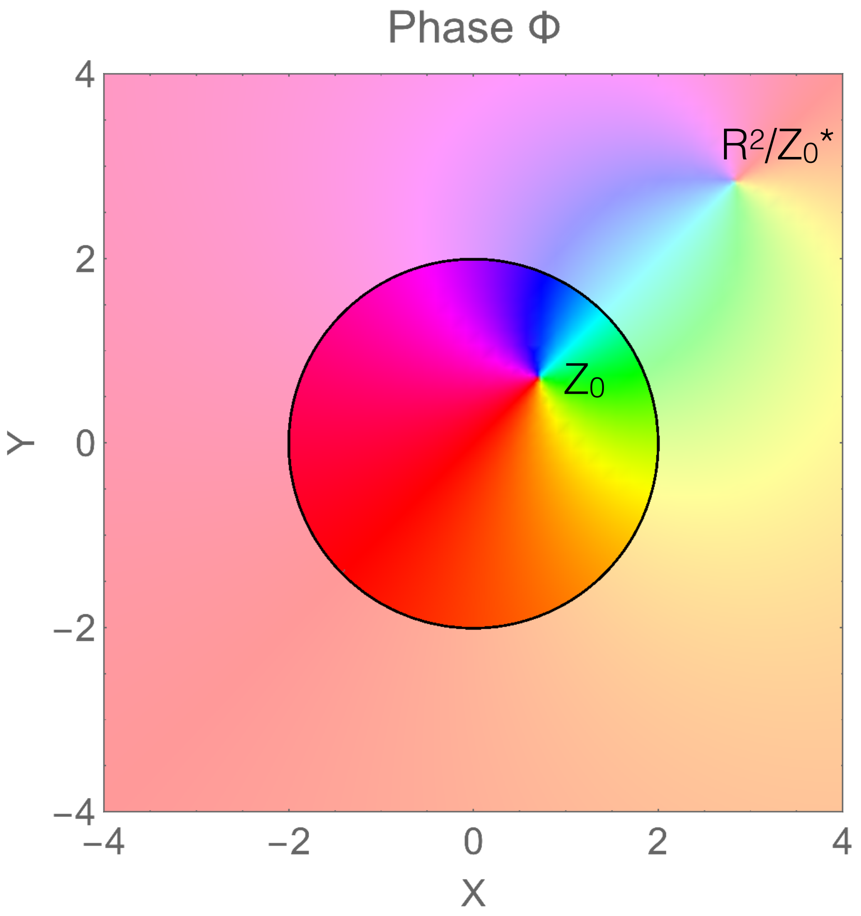
- A vortex dipole:

$$F(Z) = \log(Z - Z_1) - \log(Z - Z_2)$$



Surface with boundaries

- As in electrodynamics, use the method of images
- Single vortex on a disk of radius R: $F(Z) = \log\left(\frac{Z - Z_0}{Z - R^2/Z_0^*}\right)$



Vortex on an annulus

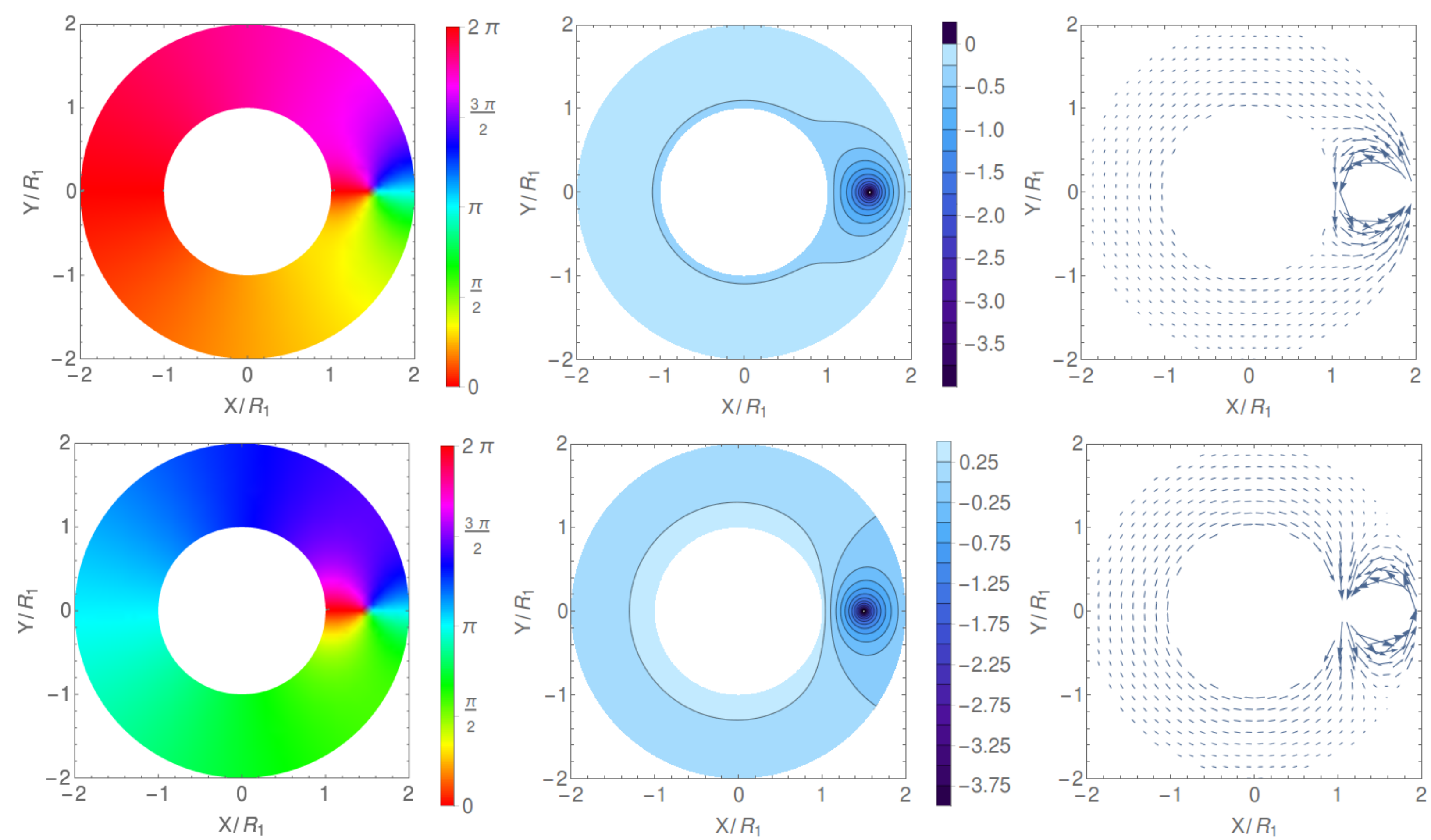
- ✦ An annulus has two boundaries → infinite series of images needed

- ✦ Potential: $F(Z) = n_1 \ln\left(\frac{Z}{R_2}\right) + \ln\left[\frac{\vartheta_1\left(-\frac{i}{2}\ln\left(\frac{Z}{Z_0}\right), \frac{R_1}{R_2}\right)}{\vartheta_1\left(-\frac{i}{2}\ln\left(\frac{ZZ_0^*}{R_2^2}\right), \frac{R_1}{R_2}\right)}\right]$

1[st] Jacobi Theta function

$n_1 = 0$

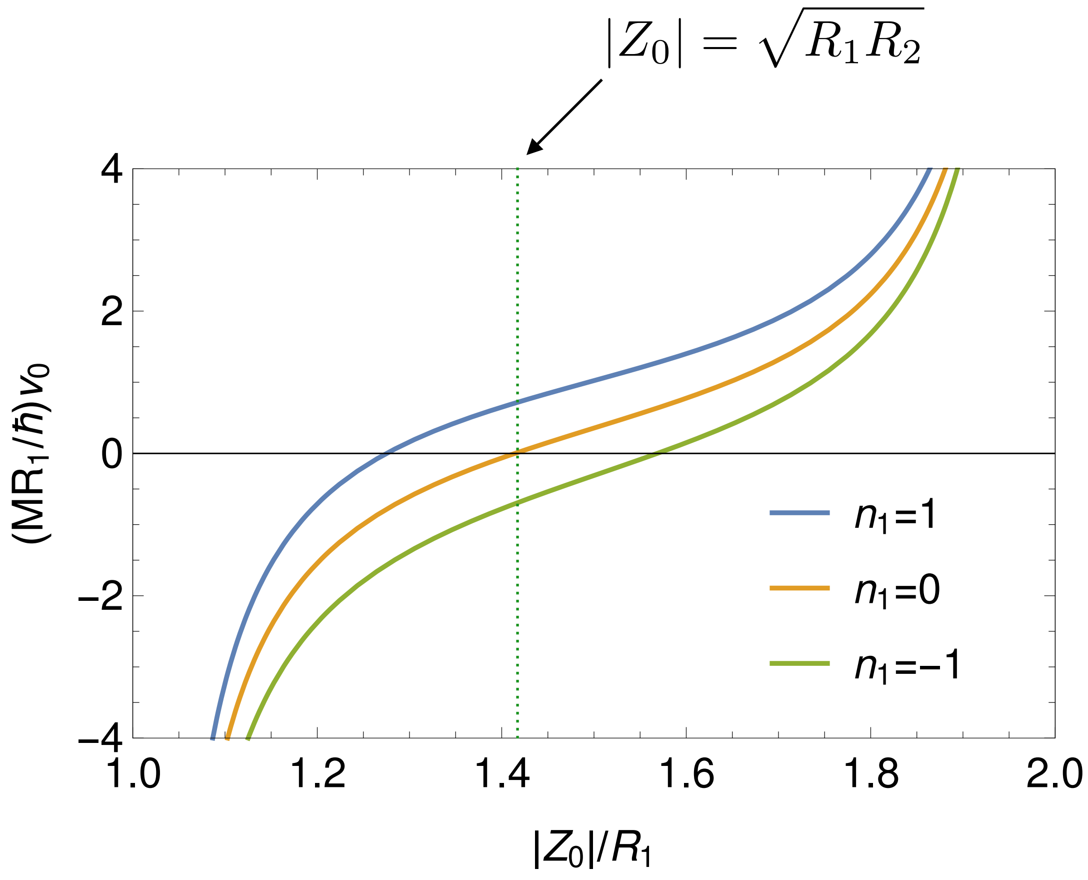
$n_1 = -1$



Velocity of the vortex core

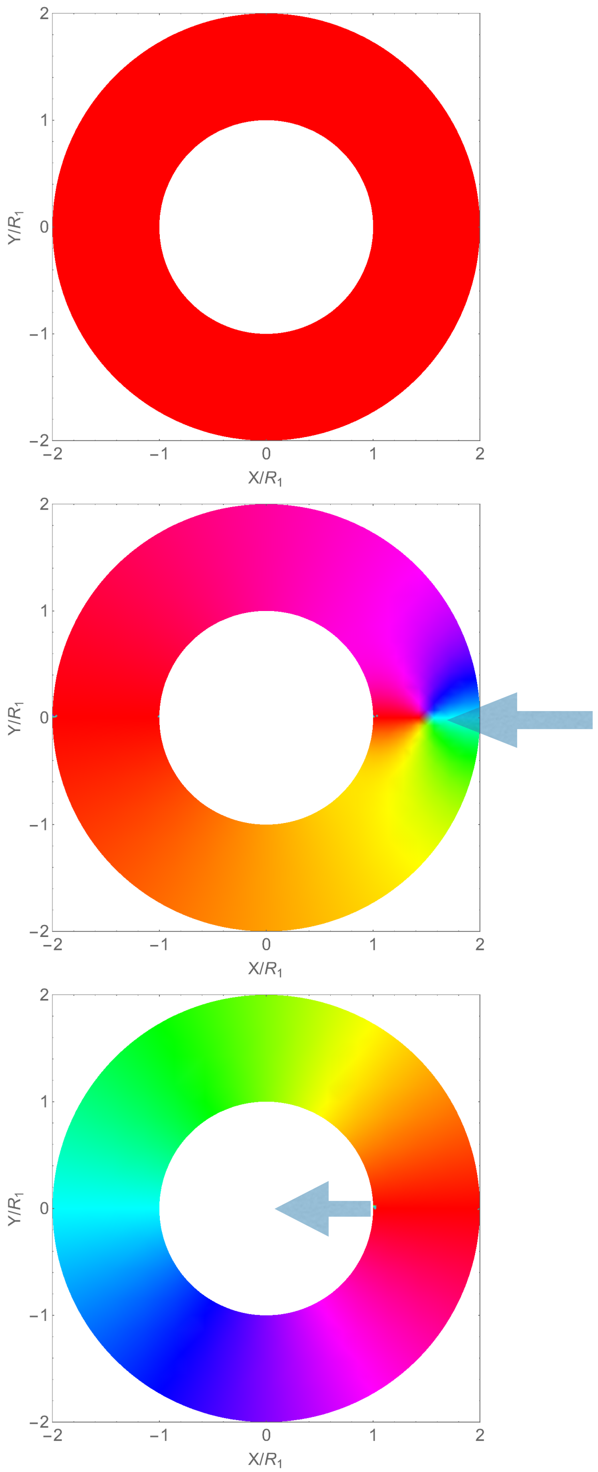
- A vortex moves with
  the local uniform flow velocity: $\dot{y}_0 + i\dot{x}_0 = \frac{\hbar}{M} \lim_{z \to z_0} \left[F'(z) - \frac{1}{z - z_0} \right]$

- Annulus with $R_2 = 2R_1$:

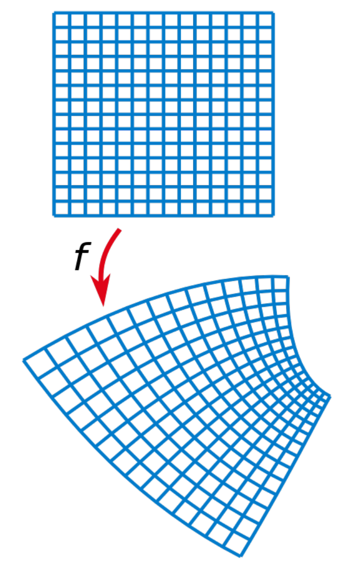


Laughlin pumping

- Start with the fluid at rest
- Stir the fluid from outside at an increasing rate
- A vortex appears on the outer edge,
  and moves inward
- The fluid (on average) rotates for $|Z| > |Z_0|$,
  but it remains stationary otherwise
- As the vortex crosses the inner edge,
  stop stirring
- The fluid is left with exactly $\hbar$ units of angular momentum per particle

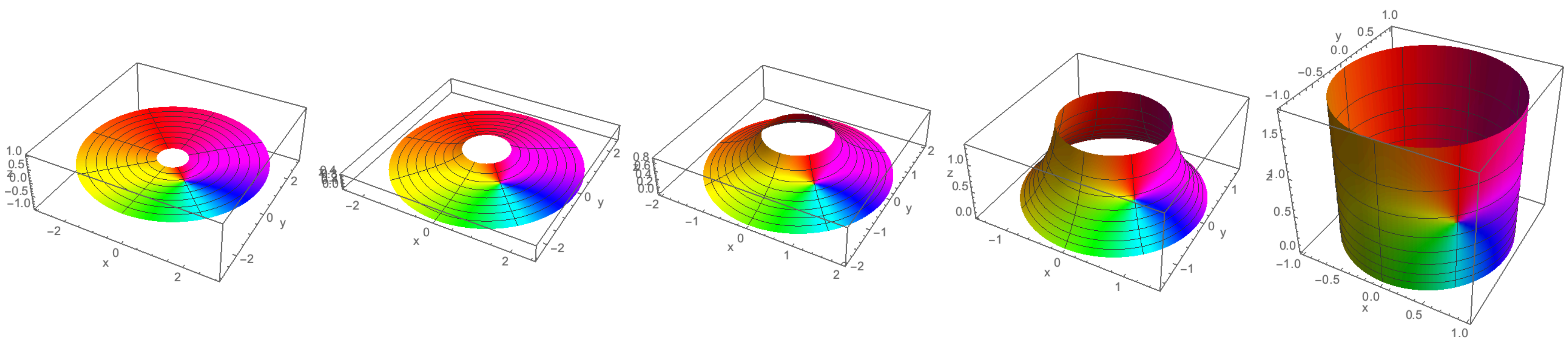


More complex surfaces?



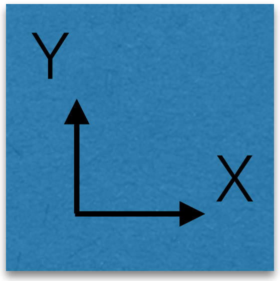
- Conformal map: $f : U \to V$ conserving angles, and shapes of infinitesimal objects

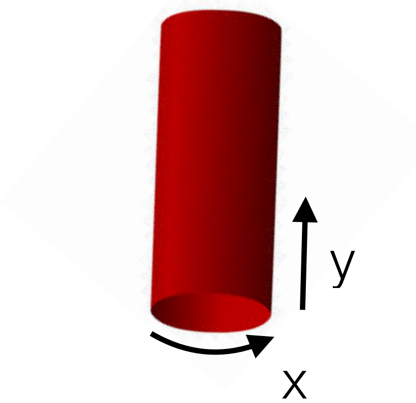
- The conformal image of a physical flow pattern is still a physical pattern



[Turner, Vitelli and Nelson, Rev. Mod. Phys. **82**, 1301 (2010)]

Vortex on a cylinder

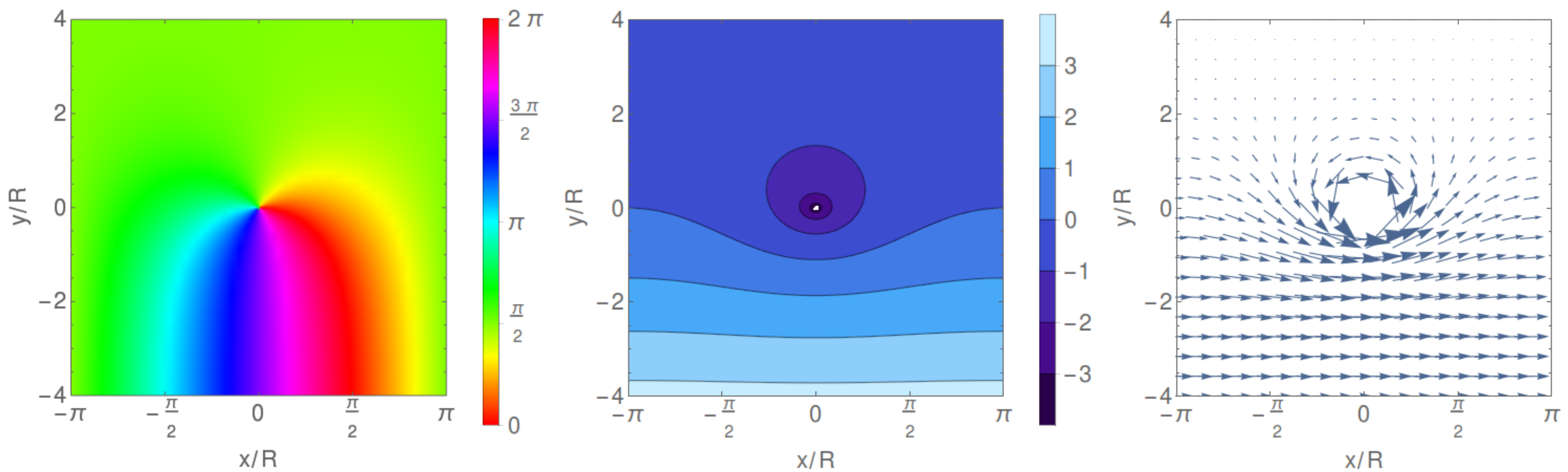




- Maps linking plane to cylinder: $Z = e^{\pm iz}$

- $F_{\text{plane}}(Z) = \ln(Z - Z_0) \quad \rightarrow \quad F_{\text{cyl},\pm}(z) = \ln(e^{\pm iz} - e^{\pm iz_0})$

$F_{\text{cyl},+}$:



- Velocity of the vortex core: $v_x = \pm \dfrac{\hbar}{2MR}$

vortices on a cylinder will *not* stand still

Energy of the fluid

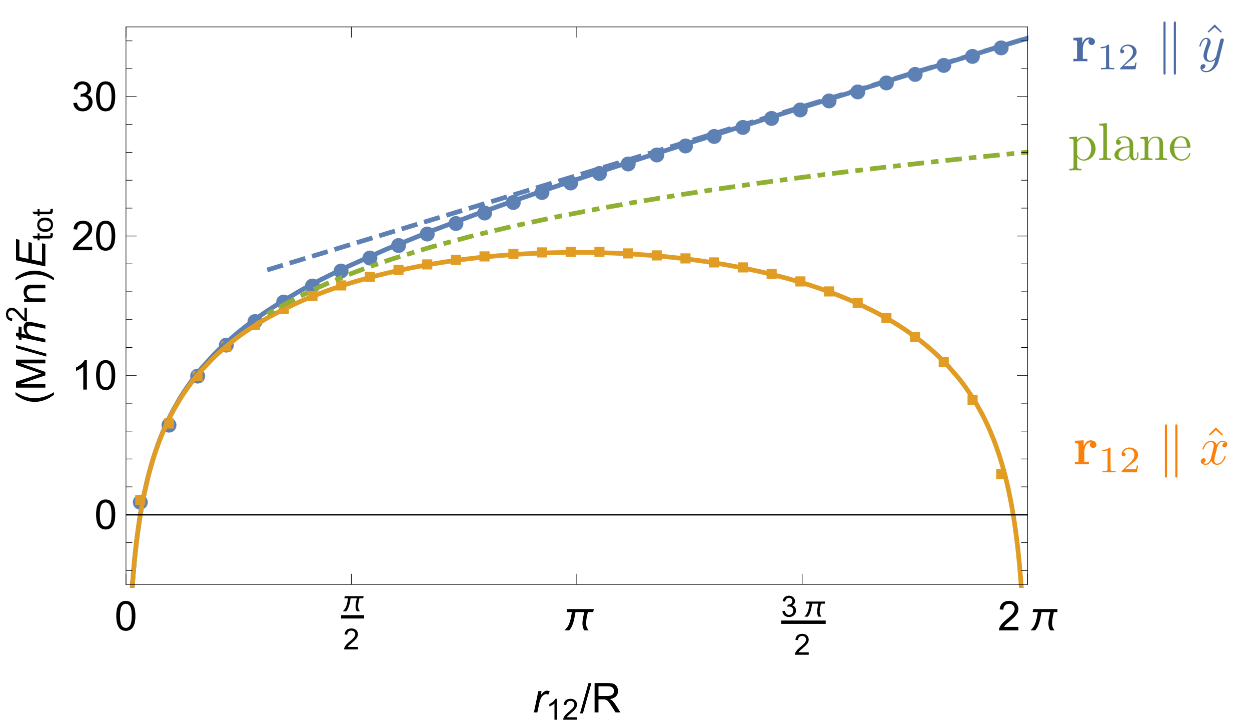
- Stream function of N vortices: $\chi(\mathbf{r}) = \sum_{i=1}^{N} q_i \chi\left(\mathbf{r} - \mathbf{r}_i\right)$

- Energy: $$E_{\text{tot}} = \frac{nM}{2} \int \mathrm{d}^2 r \, |\mathbf{v}(\mathbf{r})|^2 = \frac{n\hbar^2}{2M} \int \mathrm{d}^2 r \, |\nabla \chi(\mathbf{r})|^2 = \frac{\pi \hbar^2 n}{M} \left[N \ln\left(\frac{2R}{\xi}\right) + \sum_{i<j}^{N} q_i q_j \chi(\mathbf{r}_{ij}) \right]$$

- Energy of a vortex dipole:

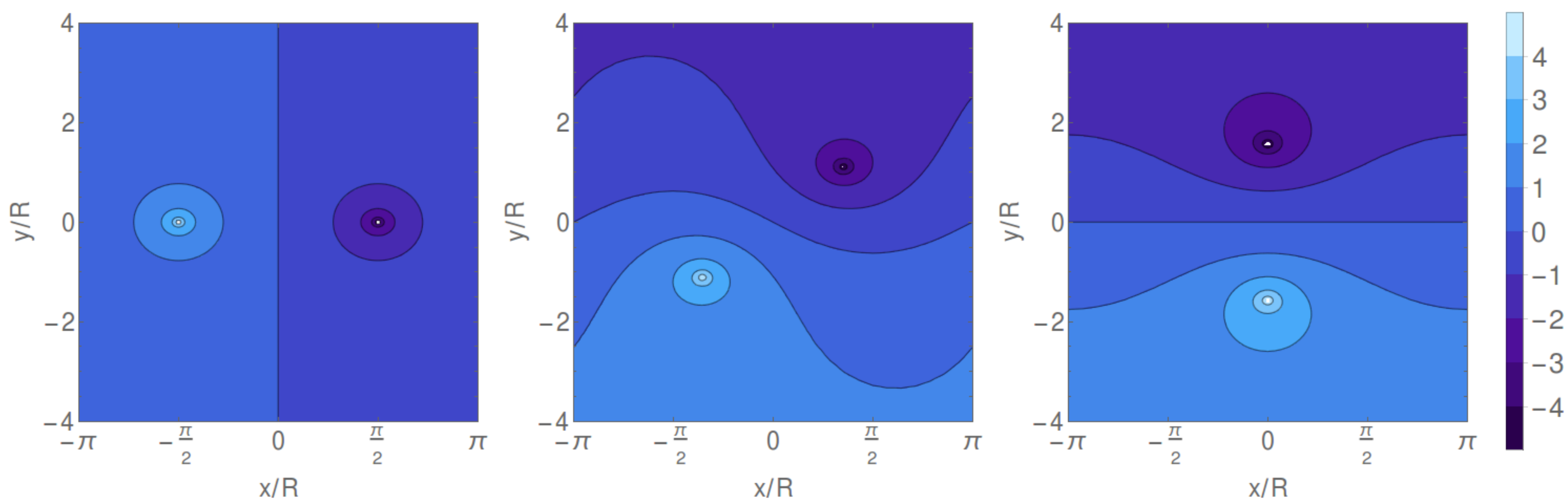
  grows linearly for $r_{12} \gg R$

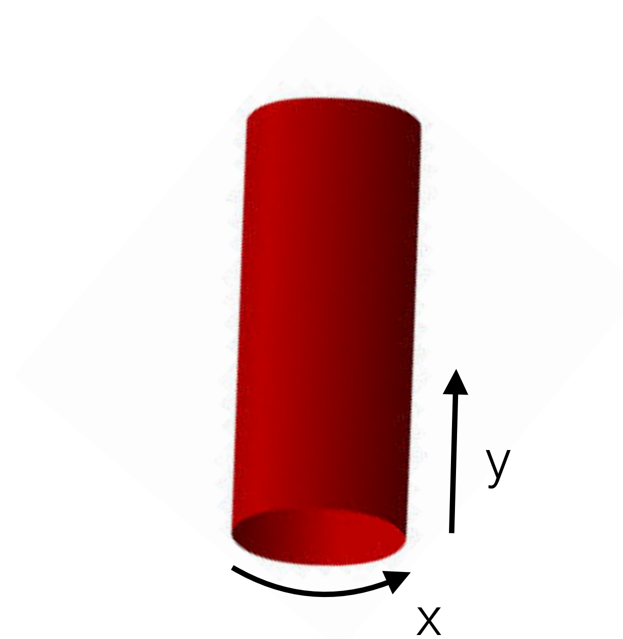
BKT physics will *not* happen on a cylinder



Motion of a vortex dipole

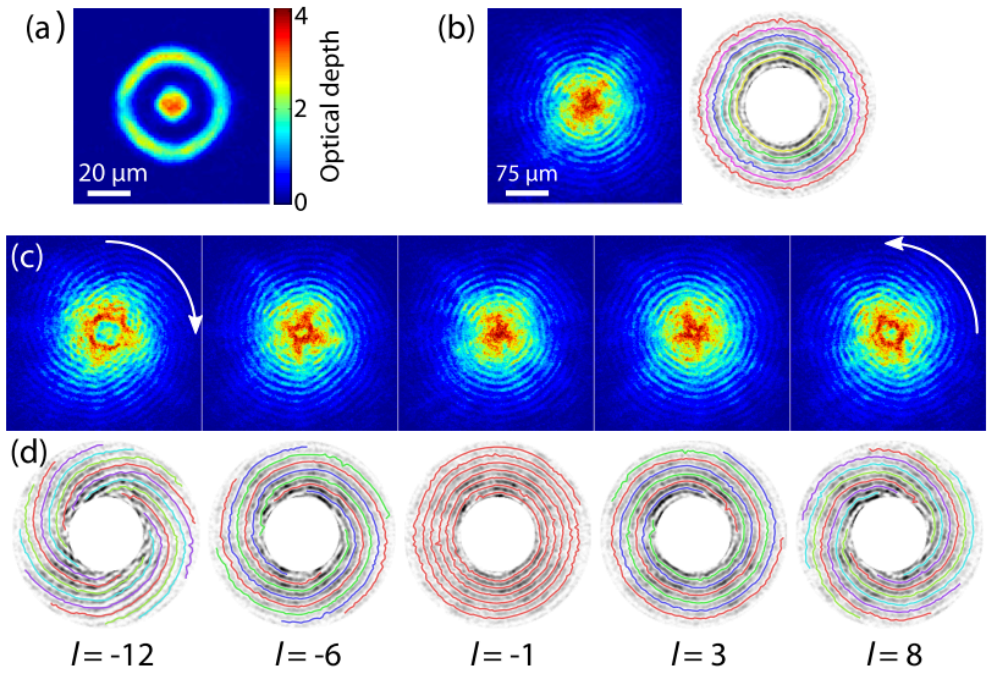
- Different trajectories, depending on the orientation of the dipole axis:





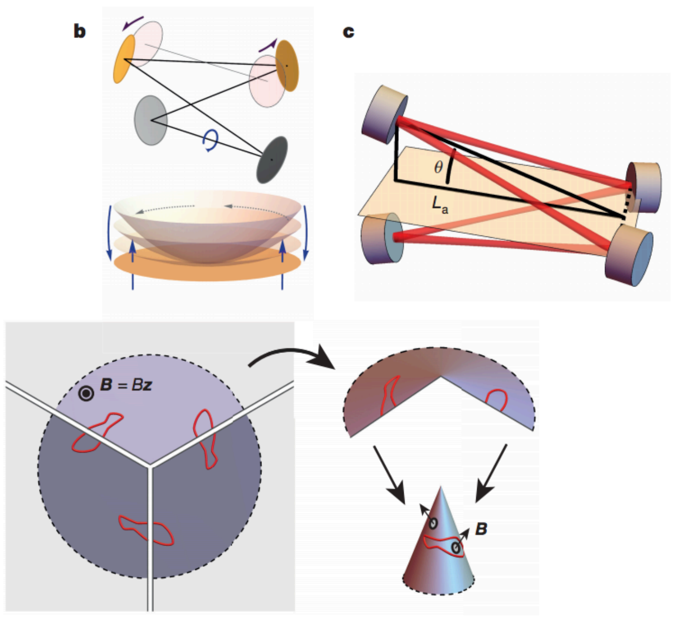
Experiments?

Ring traps for BECs:



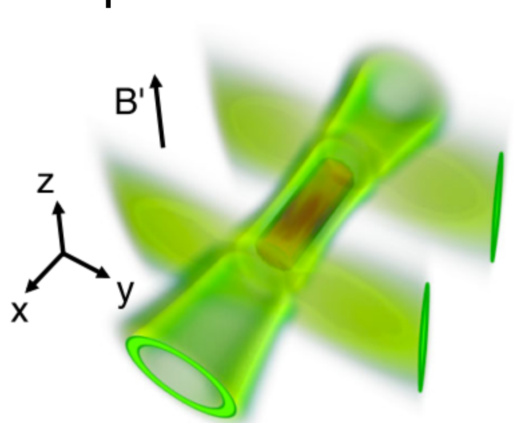
[Eckel *et al.*, Phys. Rev. X (2014)]

Twisted optical cavities:



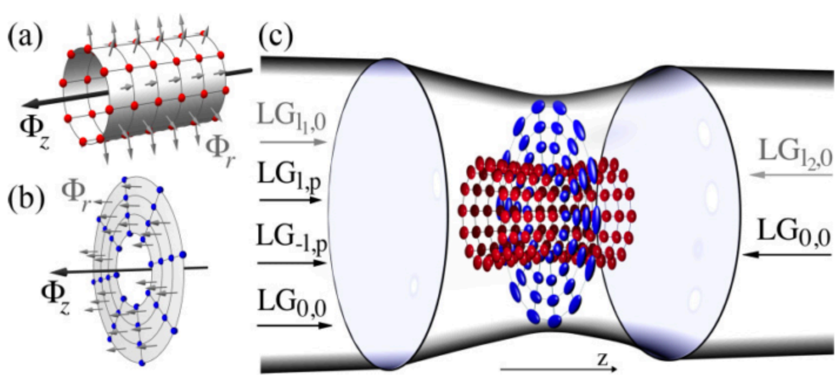
[Schine *et al.*, Nature (2016)]

Cylindrical traps for BECs:



[Gaunt *et al.*, Phys. Rev. Lett. (2013)]

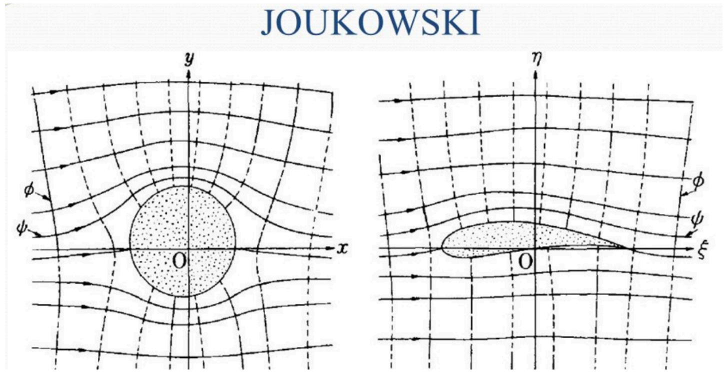
Cylindrical and annular lattices for BECs:



[Łacki *et al.*, Phys. Rev. A (2016)]

Conclusions

- *Potential flow theory* describes perfect fluids in 2D
- Images and conformal maps allow us to study peculiar geometries
- $Z_0(\Delta L_z)$: a direct hydrodynamic analog of Laughlin pumping
- On a cylinder, vortices will not stand still!
- Single-valuedness of the wave function around the cylinder imposes a quantized translational velocity to the vortex core



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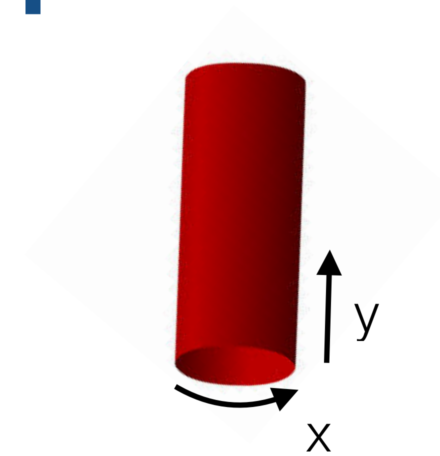
Phys. Rev. A, in press
arXiv:1708.08903

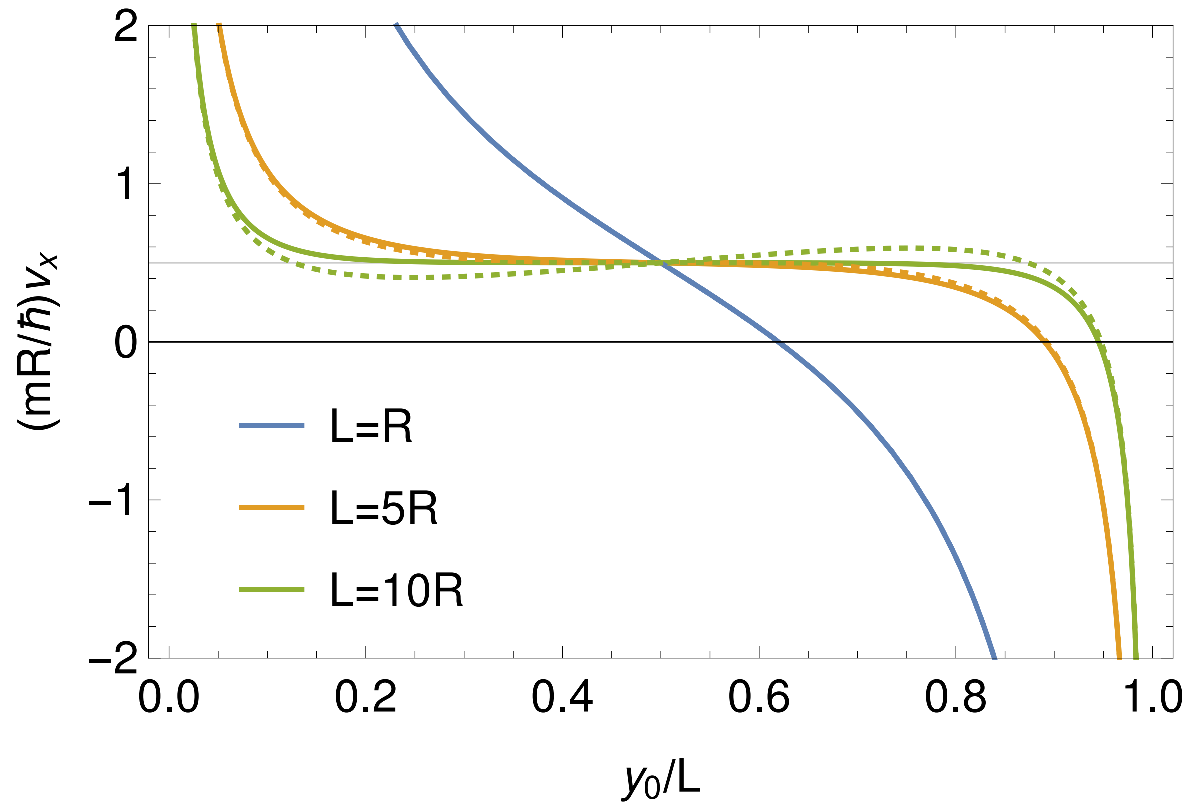
Thank you!

Vortex on a finite cylinder



- Cylinder of length L: $$F_L(z) = \ln\left[\frac{\vartheta_1\left(\frac{z-z_0}{2R}, e^{-L/R}\right)}{\vartheta_1\left(\frac{z-z_0^*}{2R}, e^{-L/R}\right)}\right]$$

- Vortex velocity:



Laughlin pumping on a cylinder

- Start with the fluid at rest
- Stir the fluid from below at a constant rate
- A vortex appears on the lower edge, and moves upward
- The fluid remains stationary above the vortex
- Below the vortex, the fluid rotates with exactly $\hbar$ units of angular momentum per particle