QUANTUM BROWNIAN MOTION IN ULTRACOLD GASES

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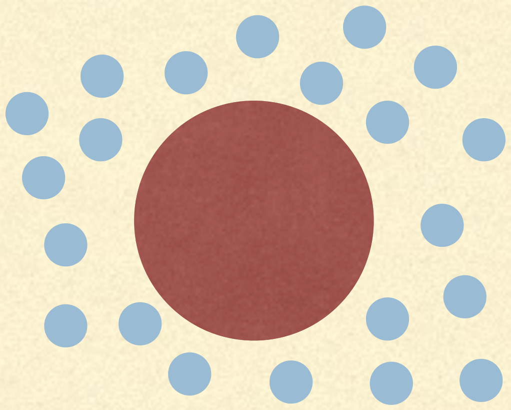
OUTLINE

- Brownian Motion
- Quantum Brownian Motion
- Born-Markov Master Equation
- Inhomogeneous environments
- Lindblad extension
- Impurities in a Bose gas

A DAMPED CLASSICAL SYSTEM

- Simplest open system: a heavy particle in a heat reservoir at temp. T, providing both friction γ and noise ξ



- Noise statistics: $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = \chi(t-t')$
- No memory (white noise): $\chi(t) \propto \delta(t)$
- "Ohmic" dissipation: time-local friction proportional to the velocity:

$$m\ddot{x}(t) + m\gamma\dot{x}(t) + V'(x) = \xi(t)$$

- To enforce equipartition theorem, $\chi(t) = 2mk_BT\gamma\delta(t)$

MEMORY EFFECTS

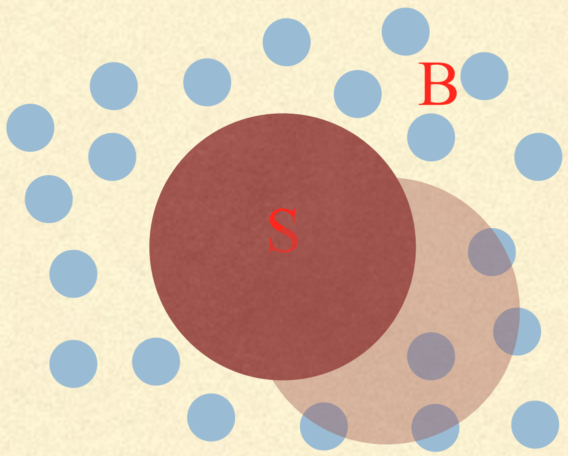
- Even if interactions are time-local in the full description, memory effects generally arise when degrees of freedom are traced out from the dynamics.

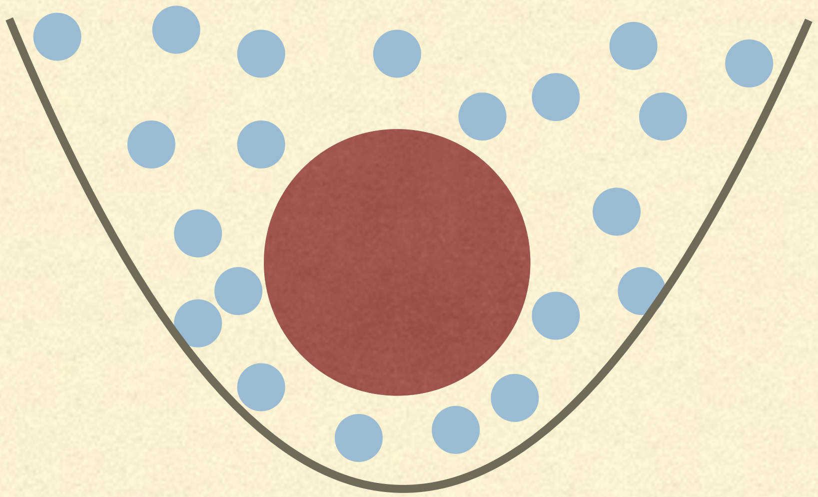
- Non-Markovian dynamics:

$$m\ddot{x}(t) + m\int_{-\infty}^{t} \mathrm{d}t' \, \gamma(t-t')\dot{x}(t') + V'(x) = \xi(t)$$

QUANTUM BROWNIAN MOTION

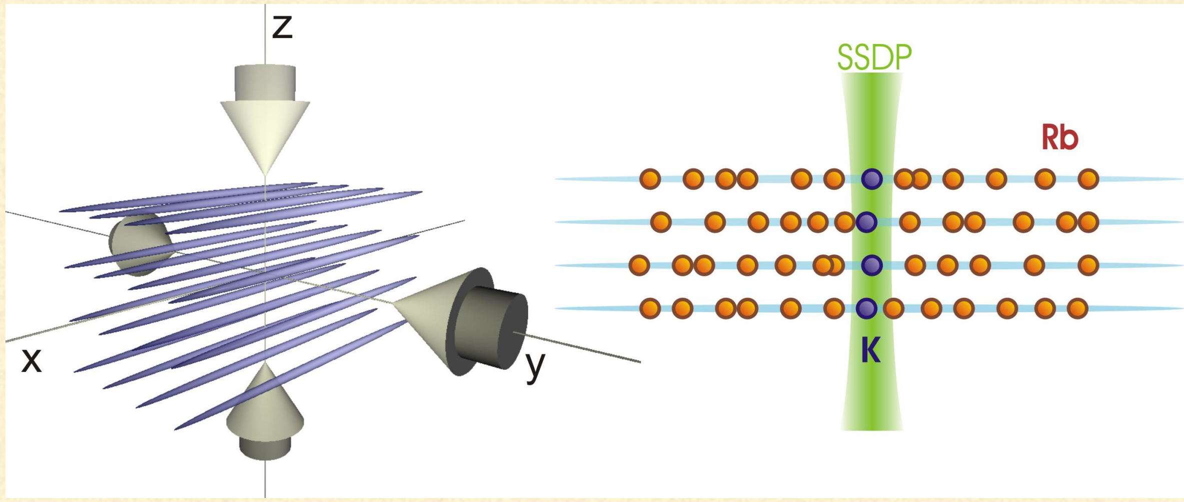
- A particle interacting with a large bath
- Microscopic origin of damping and diffusion?
- Dynamics?
- Properties of the stationary solution?
- Decoherence?
- Inhomogeneous environments?

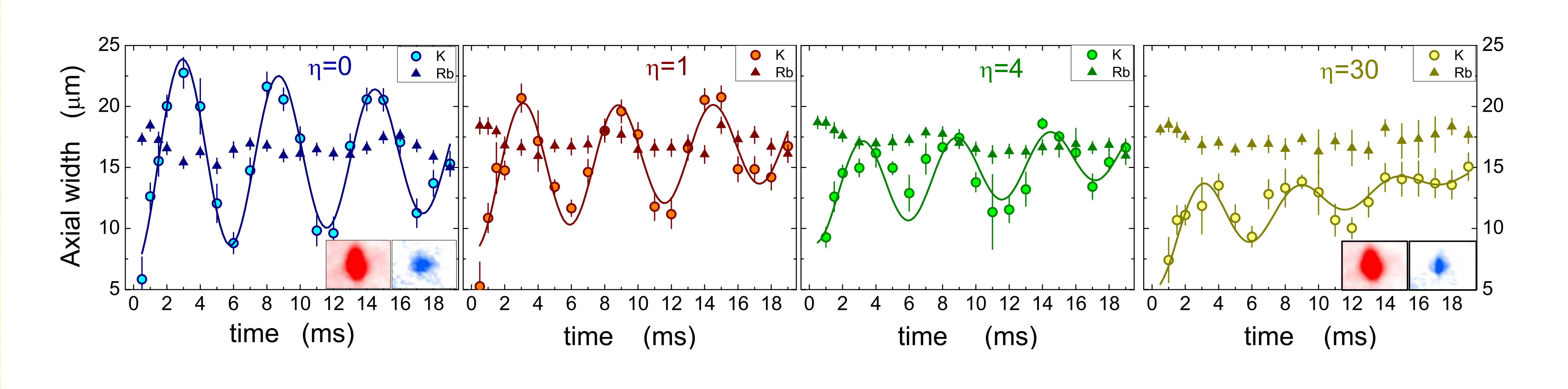




IMPURITIES IN A BOSE GAS

1D tubes:



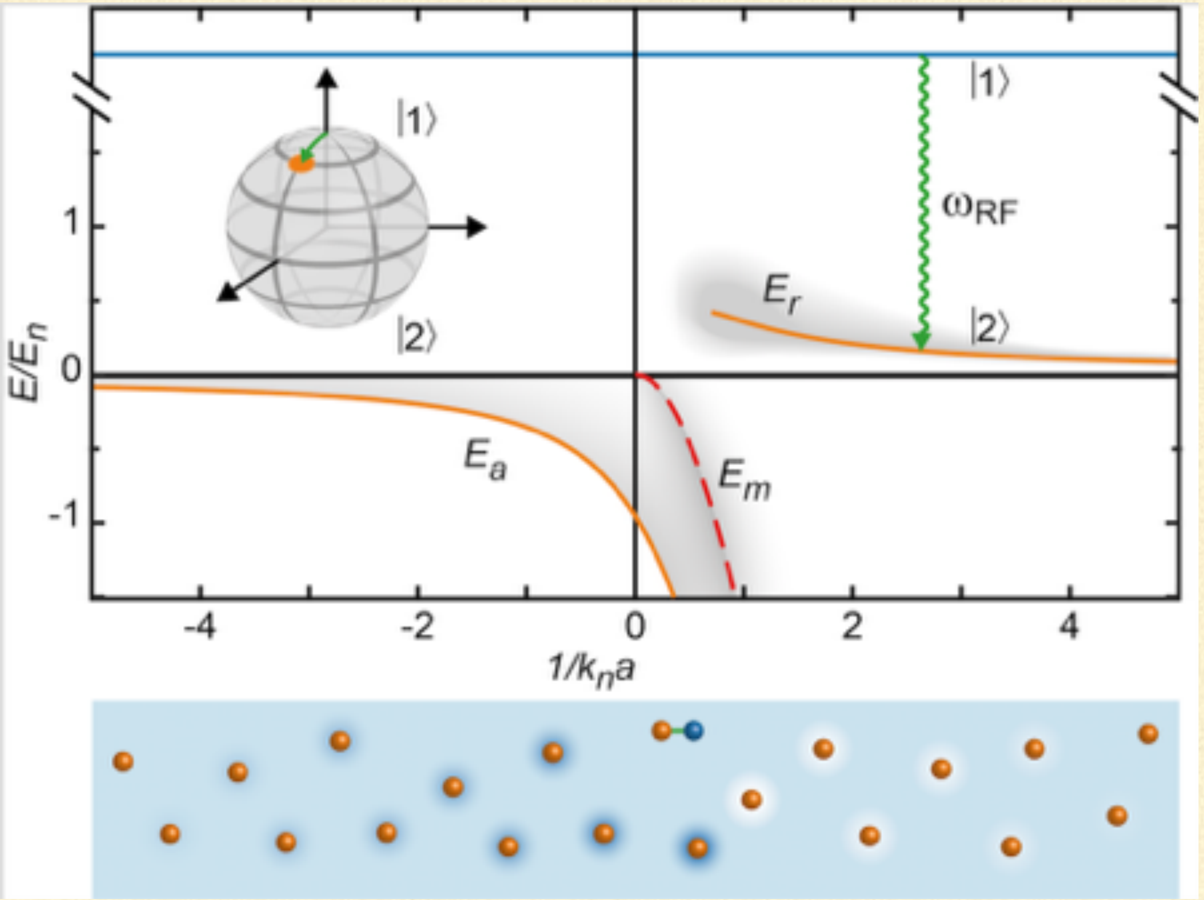


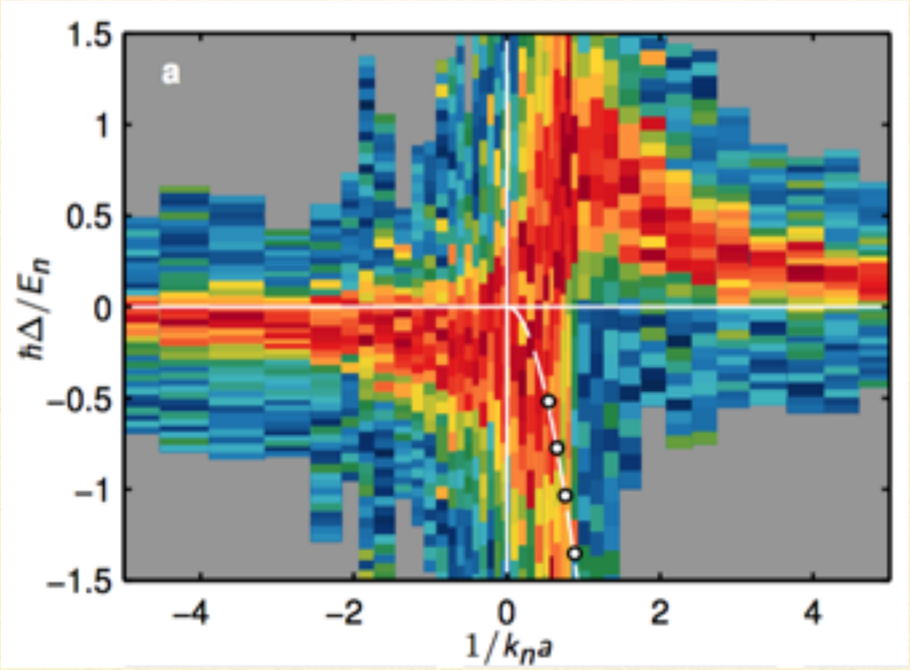
Observables: breathing frequency, damping, effective mass, …

[Catani et al., PRA (2012)]

IMPURITIES IN A BOSE GAS

Many impurities in a 3D cloud:





[Jørgensen et al, arXiv (2016); Hu et al., arXiv (2016)]

QUANTUM BROWNIAN MOTION

- A particle (S) interacting with a large bath (B): $H = H_S + H_B + H_I$

$$H_S = \frac{p^2}{2m} + V(x)$$

$$x_k = \sqrt{\frac{\hbar\omega_k}{2}}\left(g_k + g_k^\dagger\right)$$

$$H_B = \sum_k \left(\frac{p_k^2}{2m_k} + \frac{m_k\omega_k^2 x_k^2}{2}\right) - E_0 = \sum_k \hbar\omega_k g_k^\dagger g_k$$

$$H_I = -xB = -x\sum_k \kappa_k x_k$$

(homogeneous damping and diffusion)

- Spectral density: $J(\omega) = \sum_k \frac{\kappa_k^2}{2m_k\omega_k}\delta(\omega - \omega_k)$

[Caldeira and Leggett, Phys. A (1983)]

MASTER EQUATION

- Von Neumann eq. in the interaction picture: $\dot{\rho}(t) = -\frac{i}{\hbar}[H_\mathrm{I}(t), \rho(t)]$

- Formal solution: $\rho(t) = \rho(0) - \frac{i}{\hbar}\int_0^t \mathrm{d}s\,[H_I(s), \rho(s)]$

- Truncate to 2nd order, and trace over the bath:

$$\dot{\rho}_S(t) = -\frac{1}{\hbar^2}\int_0^t \mathrm{d}s\ \mathrm{Tr}_B\left([H_I(t), [H_I(s), \rho(s)]]\right)$$

- Still too complicated (the r.h.s. contains the full density matrix!)

APPROXIMATIONS

- System and bath initially uncorrelated: $\rho(0) \approx \rho_S(0) \otimes \rho_B(0)$
- Weak coupling and large environment (Born): $\rho(t) \approx \rho_S(t) \otimes \rho_B(0)$
- "Short memory" of the environment (Markov): $\tau_{\mathrm{corr}} \ll \tau_S$
- Born-Markov Master Equation:

$$\dot{\rho}_S(t) = -\frac{1}{\hbar^2} \int_{-\infty}^{t} \mathrm{d}s \, \mathrm{Tr}_B \left([H_I(t), [H_I(s), \rho_B(0) \otimes \rho_S(t)]] \right)$$

BORN-MARKOV MASTER EQUATION

- Back to Schrödinger picture:

$$\dot{\rho}_S(t) = -\frac{i}{\hbar}[H_S, \rho_S] - \frac{1}{\hbar^2}\int_0^\infty \mathrm{d}\tau\,\mathrm{Tr}_B[H_I(0), [H_I(-\tau), \rho_S(t)\otimes\rho_B(0)]]$$

- Autocorrelations of a **thermal environment:**

$$\langle g_k^\dagger g_k\rangle = \frac{1}{\exp(\omega_k/k_B T) - 1}$$

$$\mathcal{C}(\tau) = \langle B(\tau)B(0)\rangle = \sum_{k,k'} \kappa_k \kappa_{k'} \langle x_k(\tau) x_{k'}(0)\rangle \delta_{k,k'}$$

$$= \int_0^\infty \mathrm{d}\omega\; J(\omega)\left[\coth\left(\frac{\omega}{2k_B T}\right)\cos(\omega\tau) - i\sin(\omega\tau)\right]$$

$$= \nu(\tau) + i\eta(\tau)$$

noise dissipation

BORN-MARKOV MASTER EQUATION

($\rho_S \to \rho$ in the following)

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H_S, \rho(t)] - \frac{1}{\hbar^2}\int_0^\infty \mathrm{d}\tau \left(\nu(\tau)[x(0),[x(-\tau),\rho(t)]] - i\eta(\tau)[x(0),\{x(-\tau),\rho(t)\}]\right)$$

- **Harmonically trapped central particle:** $V(x) = \frac{1}{2}m\Omega^2 x^2$
- $x(\tau) = x\cos(\Omega\tau) + (p/m\Omega)\sin(\Omega\tau)$
- $\dot{\rho}(t) = -\frac{i}{\hbar}\left[H_S + C_x x^2, \rho(t)\right] - \frac{iC_p}{\hbar m\Omega}[x,\{p,\rho(t)\}] - \frac{D_x}{\hbar}[x,[x,\rho(t)]] - \frac{D_p}{\hbar m\Omega}[x,[p,\rho(t)]]$

$$C_x = -\int_0^\infty \mathrm{d}\tau\, \eta(\tau)\cos(\Omega\tau) \qquad D_x = \int_0^\infty \mathrm{d}\tau\, \nu(\tau)\cos(\Omega\tau)$$

$$C_p = \int_0^\infty \mathrm{d}\tau\, \eta(\tau)\sin(\Omega\tau) \qquad D_p = -\int_0^\infty \mathrm{d}\tau\, \nu(\tau)\sin(\Omega\tau)$$

- **Decoherence rate:** $-\frac{D_x}{\hbar}[x,[x,\rho(t)]] \longrightarrow -\gamma\left(\frac{X - X'}{\lambda_{\mathrm{dB}}}\right)^2 \rho(X, X', t)$

WIGNER REPRESENTATION

Probabilities:

$$\int_{-\infty}^{\infty} \mathrm{d}p\, W(x,p) = \langle x|\rho|x\rangle$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, W(x,p) = \langle p|\rho|p\rangle$$

- Wigner function (quasi-probability):

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \langle x+y|\rho|x-y\rangle e^{-2ipy/\hbar}$$

$$\dot{W} = \left[m\Omega^2\partial_p x - \frac{\partial_x p}{m} + \frac{2C_p}{m\Omega}\partial_p p + \hbar D_x \partial_p^2 - \frac{\hbar D_p}{m\Omega}\partial_x\partial_p\right] W$$

- Ansatz for the stationary solution: $W \propto \exp\left[-\left(c_p \frac{p^2}{2m} + c_x \frac{m\Omega^2 x^2}{2}\right)/(k_B\tilde{T})\right]$

STATIONARY STATE

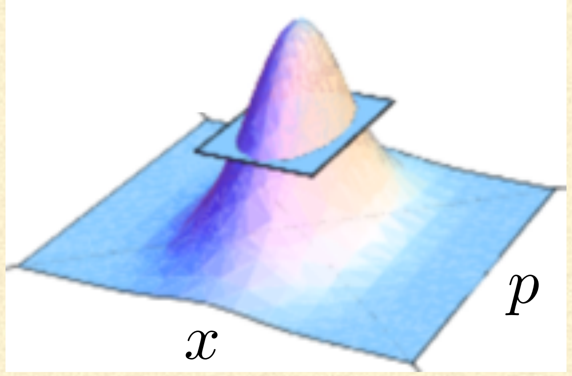
$$\delta_x = 2\sqrt{\frac{m\Omega^2 \langle x^2 \rangle_{\rm st}}{2\hbar\Omega}}$$

$$\delta_p = 2\sqrt{\frac{\langle p^2 \rangle_{\rm st}}{2m\hbar\Omega}}$$

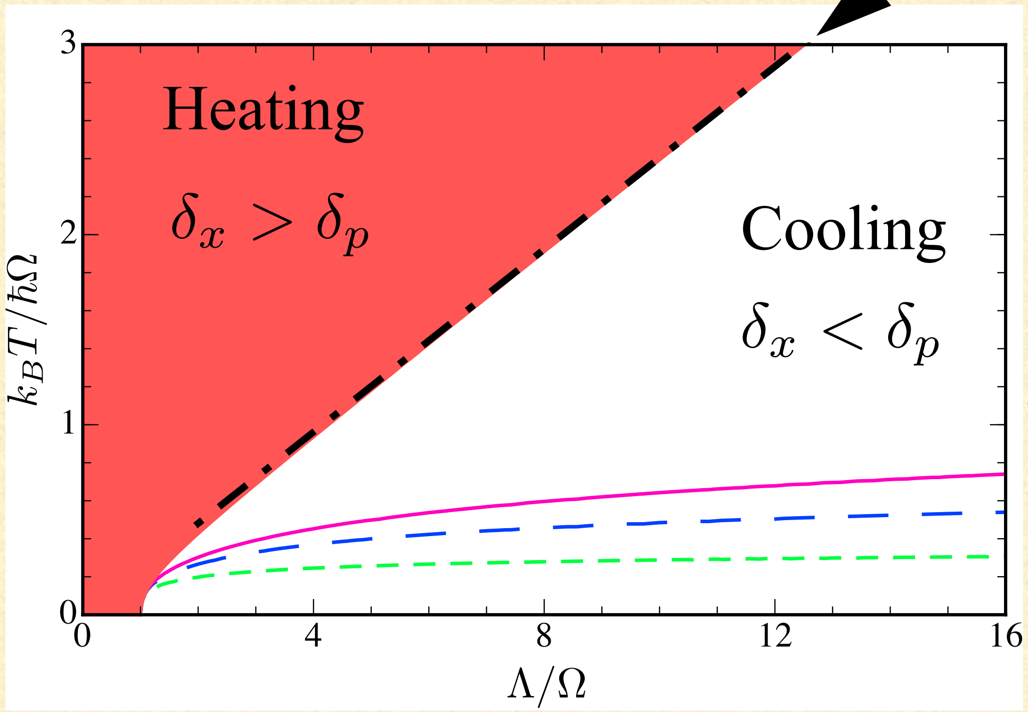
Eff. area of the W-distrib.: $\delta_x \delta_p$ vs. $\coth\left(\frac{\hbar\Omega}{2k_B T}\right)$

Squeezing: δ_x vs. δ_p

Heisenberg uncertainty: $\delta_x \delta_p \geq 1$



High-temperature limit $k_B T/\hbar \gg \Lambda \gg \Omega$: Gibbs

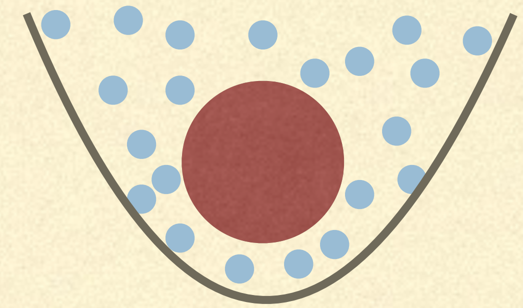


spectral density: Ohmic + Lorentz-Drude

$$J(\omega) = \frac{m\gamma}{\pi}\omega\frac{\Lambda^2}{\omega^2 + \Lambda^2}$$

[Massignan et al., PRA (2015)]

INHOMOGENEOUS ENVIRONMENTS



- How to treat inhomogeneous damping and diffusion?

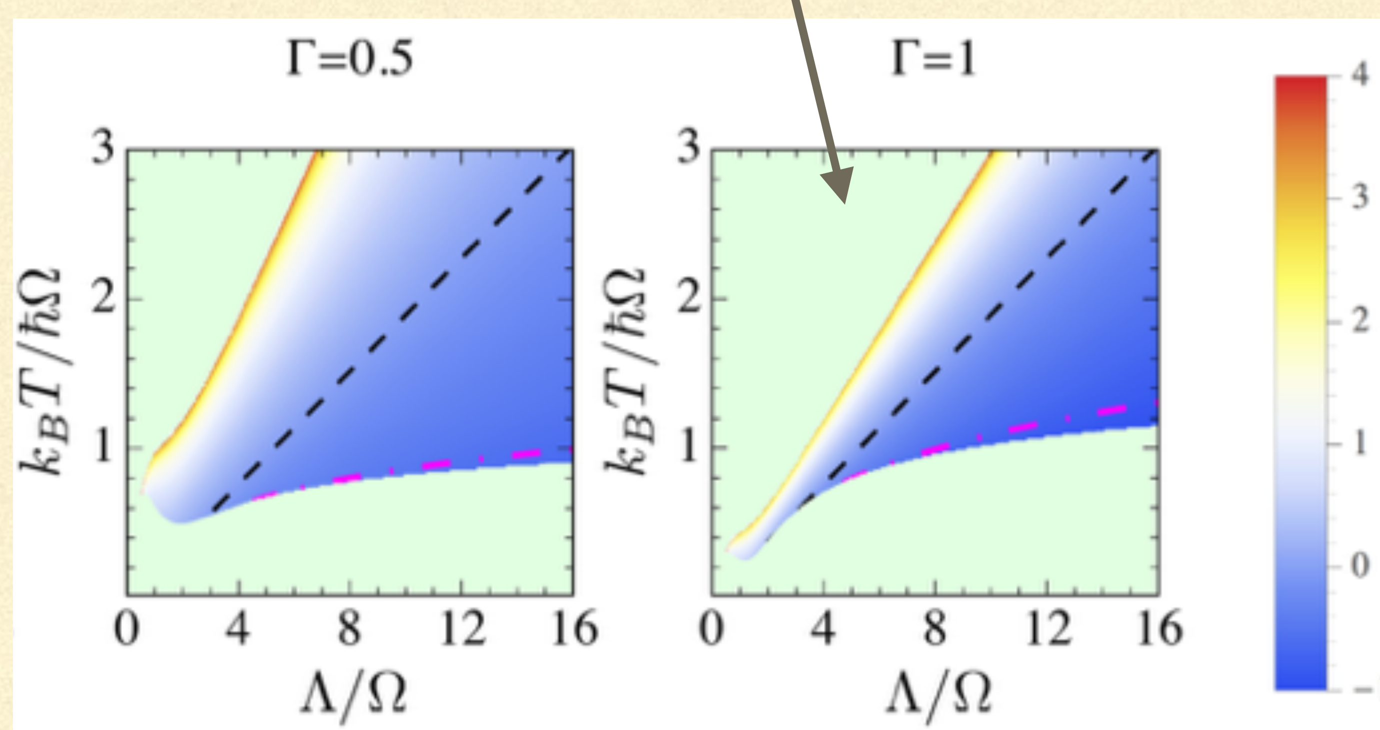
$$H_I = -xB = -x \sum_k \kappa_k x_k \quad \longrightarrow \quad H_I = -f(x)B$$

- Quadratic coupling: $f(x) = x^2/a$

large Lamb shift of the trapping frequency → instabilities

Aspect ratio: $\log(\delta_x^2/\delta_p^2)$

$$\Gamma = \frac{2\hbar\gamma}{m\Omega^2 a^2}$$



[Massignan et al., PRA (2015)]

HEISENBERG VIOLATIONS?

- The Born-Markov Master Equation for QBM $\dot{\rho}_S(t) = \mathcal{L}\rho_S(t)$ yields violations of the Heisenberg uncertainty relation at low T!

- ∃ exact solution, showing that: $\dot{\rho}_S(t) = \mathcal{L}(t)\rho_S(t)$

  [strictly speaking, QBM is <u>not</u> a Markov process]

- Mathematical reason:

  the QBM BMME does not preserve the positivity of ρ

HEISENBERG UNCERTAINTY FOR PURE AND MIXED STATES

- Pure states: $\sigma_A^2 = \left\langle \psi \middle| (A - \langle A \rangle)^2 \middle| \psi \right\rangle$ ➡ $\sigma_X^2 \sigma_P^2 \geq \frac{1}{4} \left\langle \frac{[X,P]}{i} \right\rangle^2 = \frac{\hbar^2}{4}$

$$|a\rangle \equiv \left| (A - \langle A \rangle)\psi \right\rangle \quad \longrightarrow \quad \langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2 = \left(\frac{\langle a|b\rangle + \langle b|a\rangle}{2}\right)^2 + \left(\frac{\langle a|b\rangle - \langle b|a\rangle}{2i}\right)^2 \geq \left(\frac{\langle a|b\rangle - \langle b|a\rangle}{2i}\right)^2 = \left\langle \frac{[A,B]}{2i} \right\rangle^2$$

- Density matrices: $\sigma_A^2 = \mathrm{Tr}\left[\rho\,(A - \langle A \rangle)^2\right]$ ➡ $\sigma_X^2 \sigma_P^2 \geq \frac{1}{4} \left\langle \frac{[X,P]}{i} \right\rangle^2 = \frac{\hbar^2}{4}$

but the proof requires $\rho = \sum_j p_j \, |\phi_j\rangle \langle \phi_j|$ with $p_j > 0$

LINDBLAD EQUATION

- $\langle\psi|\rho(t)|\psi\rangle > 0$ ensured by: $\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{i,j} \kappa_{ij} \left[A_i \rho(t) A_j^\dagger - \frac{1}{2}\{A_i^\dagger A_j, \rho(t)\} \right]$

  [time-local & Markovian]

  (κ_{ij}): positive definite matrix

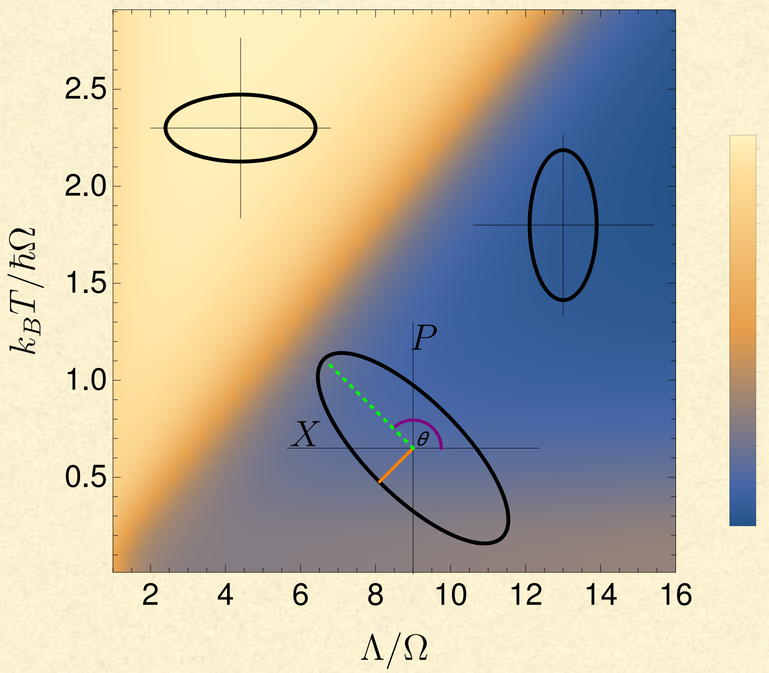
- Minimal choice: single Lindblad generator: $A = \alpha x + \beta p$

$$\dot{\rho}(t) = -\frac{i}{\hbar}\left[H_S + C_x x^2, \rho(t)\right] - \frac{iC_p}{\hbar m \Omega}[x, \{p, \rho(t)\}] - \frac{D_x}{\hbar}[x, [x, \rho(t)]] - \frac{D_p}{\hbar m \Omega}[x, [p, \rho(t)]] - \frac{\tilde{D}_p}{\hbar m^2 \Omega^2}[p, [p, \rho(t)]]$$

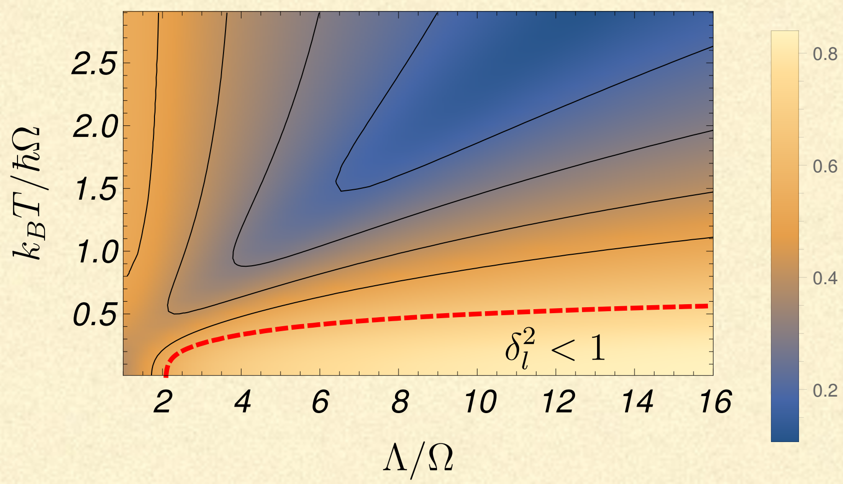
- Ansatz: correlated Gaussian $W \propto \exp\left[\frac{1}{2(\rho^2 - 1)}\left(\frac{x^2}{\sigma_x^2} + \frac{p^2}{\sigma_p^2} + \frac{2\rho x p}{\sigma_x \sigma_p}\right)\right]$

STATIONARY STATE OF THE LINDBLAD-BMME FOR QBM

correlations
and squeezing

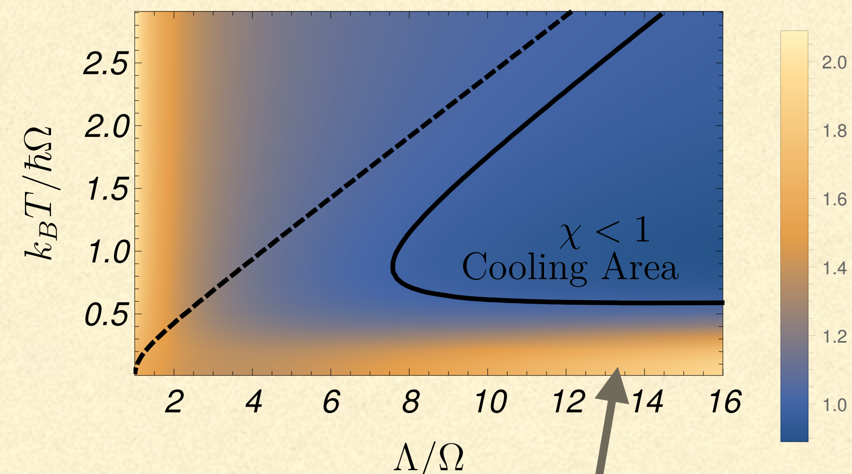


eccentricity



$$\eta = \sqrt{1 - (\delta_l/\delta_L)^2}$$

heating/cooling



$$\chi = \frac{\delta_l \delta_L}{\coth\left(\frac{\hbar\Omega}{2k_BT}\right)}$$

[Lampo, Massignan et al., arXiv (2016)]

IMPURITIES IN A BOSE GAS

$$\hat{H}_B = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k'},\mathbf{q}} V_B(\mathbf{q}) \hat{a}^\dagger_{\mathbf{k'}-\mathbf{q}} \hat{a}^\dagger_{\mathbf{k}+\mathbf{q}} \hat{a}_{\mathbf{k'}} \hat{a}_{\mathbf{k}}$$

$$\hat{H}_{IB} = \frac{1}{V} \sum_{\mathbf{k},\mathbf{q}} V_{IB}(\mathbf{k}) \hat{\rho}_I(\mathbf{q}) \hat{a}^\dagger_{\mathbf{k}-\mathbf{q}} \hat{a}_{\mathbf{k}}$$

Bogolubov transformation: $\hat{H}_{IB} = \sum_{\mathbf{k}\neq\mathbf{0}} V_{\mathbf{k}} e^{i\mathbf{k}\cdot\hat{\mathbf{r}}} \left(\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\right)$

$$V_{\mathbf{k}} = g_{IB} \sqrt{\frac{n_0}{V}} \left[\frac{(\xi k)^2}{(\xi k)^2 + 2}\right]^{\frac{1}{4}}$$

Lee-Low-Pines transformation:

$$\hat{H} = \frac{\left(\hat{\mathbf{p}} - \sum_{\mathbf{k}} \mathbf{k} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\right)^2}{2m_I} + \frac{m_I \Omega^2 \hat{\mathbf{r}}^2}{2} + \sum_{\mathbf{k}\neq\mathbf{0}} E_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \sum_{\mathbf{k}\neq\mathbf{0}} V_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\right)$$

[Shashi et al., PRA (2014)]

IN COLLABORATION WITH:

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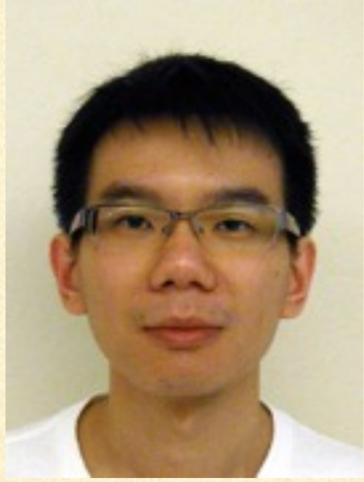

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