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Strong-coupling ansatz for the one-dimensional Fermi gas in a harmonic potential

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Abstract

The 1D Fermi gas with repulsive contact interactions provides an important model of strong correlations and is often amenable to exact methods. However, in the presence of confinement there is no exact solution for an arbitrary number of interacting fermions. We propose [1] a novel scheme for generating the lowest-energy wavefunctions of the 1D Fermi gas in a harmonic potential near the Tonks-Girardeau limit. We specialize to the case of a single ↓ particle interacting with N ↑ particles. Comparing with exact numerics, we show that the wavefunction overlap of all states in the ground state manifold obtained with our ansatz exceeds 0.9997 for N≤8. Our approximation yields an effective Heisenberg model which we solve combinatorially for arbitrary N, with a fidelity for the absolute ground state extrapolating to 0.9999 as N→∞. The Heisenberg model may be extended to an arbitrary number of spin ↑, ↓ particles, and to a 2-component Bose gas with resonant inter- and intraspecies interactions [2].

One-dimensional fermions in a harmonic potential

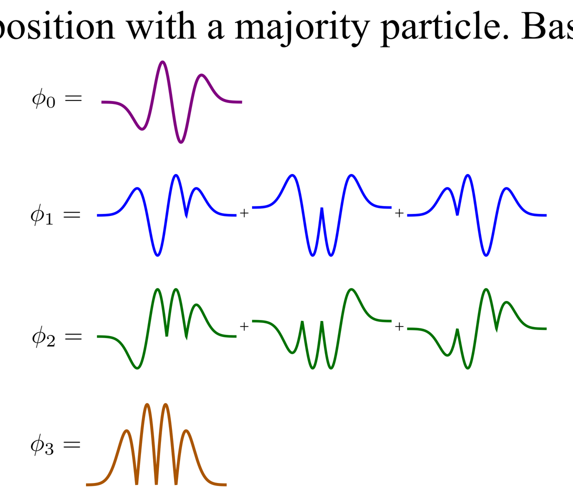
- Model Hamiltonian for $N\uparrow$ fermions and $1\downarrow$ particle at x_0

$$\mathcal{H}=\sum_{i=0}^{N}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}m\omega^2x_i^2\right]+g\sum_{i<j}\delta(x_i-x_j)$$

- In the TG limit, there are N+1 degenerate states, corresponding to the number of unique configurations of the impenetrable particles. The simplest is the fully ferromagnetic state with spin $S=(N+1)/2$

$$\psi_0(\mathbf{x})=\mathcal{N}_N\left(\prod_{0\le i<j\le N}x_{ij}\right)e^{-\sum_{k=0}^{N}x_k^2/2}$$

- We construct the remaining states with $S=S_z=\frac{N-1}{2}$ as superpositions of basis functions where we replace 1,…,N zero-crossings with cusps when the impurity exchanges position with a majority particle. Basis functions for N=3:



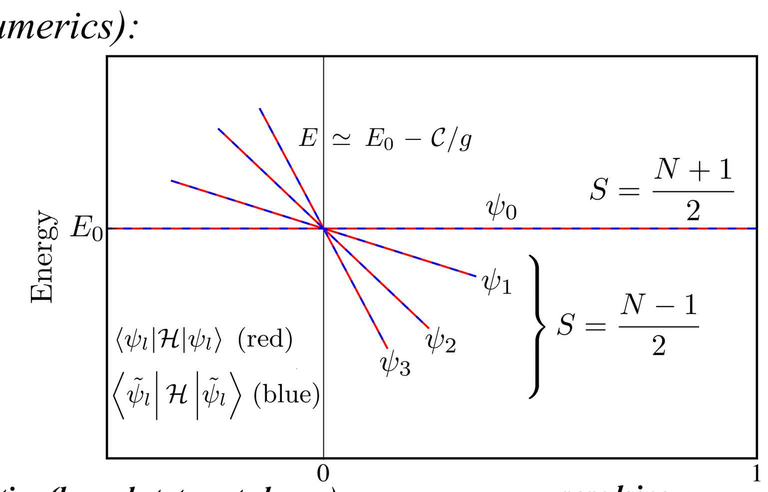
Ansatz

For any N, the l'th wavefunction is a superposition of basis functions with at most l cusps inserted

- ***Allowing cusps in the wavefunction leads to a lower energy, as these may be easily relaxed at finite repulsion.***
- ***The eigenstates can be found through Gram-Schmidt orthogonalization on the set of basis states. This procedure is easily carried out, as it is essentially combinatorial.***

Comparison with exact numerics

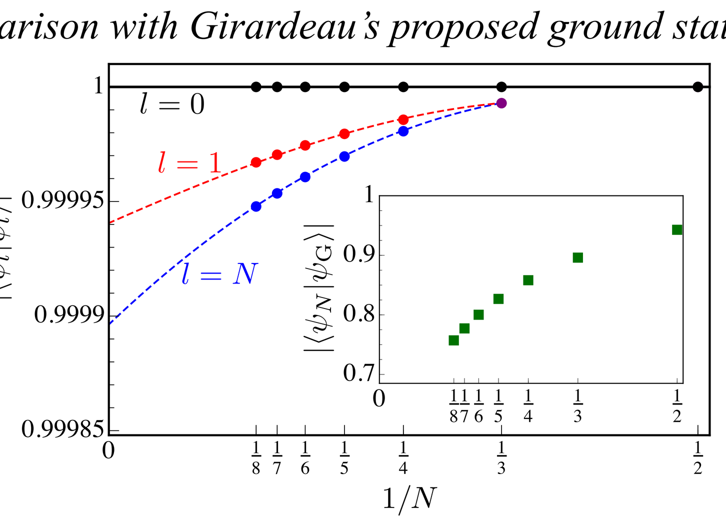
Illustration of the ground state manifold for N=3 (red=ansatz, blue=exact numerics):



Comparison with exact numerics

We compare with exact numerics for N≤8 and find that all overlaps exceed 0.9997. ***The ansatz is essentially an exact solution to the problem.***

Overlaps between exact and ansatz states for the highest energy states (l=0 and l=1), and the ground state (l=N). Inset: Comparison with Girardeau's proposed ground state



Spectrum

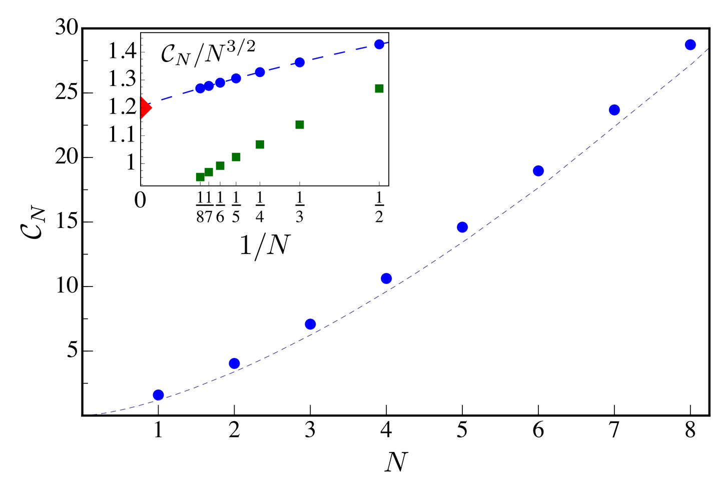
The energy of the l'th state in the vicinity of the TG limit is

$$E_l\simeq E_0-\mathcal{C}_l/g$$

We find a spectrum reminiscent of the L^2 angular momentum operator:

$$\mathcal{C}_l\approx\mathcal{C}_N\frac{l(l+1)}{N(N+1)}$$

We also find that the ground state contact coefficient is consistent with McGuire's Bethe ansatz solution, properly generalized to the harmonic potential [5] using LDA



Effective Heisenberg model

In the limit g→∞, the system effectively consists of impenetrable particles since the wavefunction must vanish when two particles approach each other. This allows us to consider the system in the TG limit as an effective lattice, with the particle to the left at site i = 0, and so on. A small finite value of 1/g introduces an effective nearest neighbor spin interaction, governed by a Heisenberg XXX model [6,7]

$$\mathcal{H}=E_F+\frac{\mathcal{C}_N}{g}\left[\sum_{i=0}^{N-1}J_i\mathbf{S}^i\cdot\mathbf{S}^{i+1}-\frac{1}{4}\sum_{i=0}^{N-1}J_i\right]$$

- Within our ansatz we evaluate the exchange coefficients combinatorially *for any N*

$$J_i=\frac{-\left(i-\frac{N-1}{2}\right)^2+\frac{1}{4}(N+1)^2}{N(N+1)/2}$$

- We solve this model analytically for any N, the solutions being the discrete Chebyshev polynomials. The ground state takes the form of a sign-alternating Pascal's triangle:

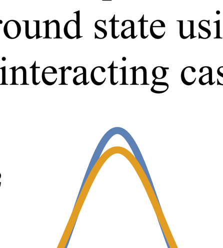
$$\left|\tilde\psi_N\right\rangle\propto\sum_{i=0}^{N}(-1)^iS_-^i\left|\uparrow\cdots\uparrow\right\rangle$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Pascal

- For N→∞ we evaluate the probability distribution of the impurity atom in the ground state using LDA. This is almost the same as in the non-interacting case:

$$P(x_0)\propto e^{-8x_0^2/\pi^2}\qquad g=\infty$$



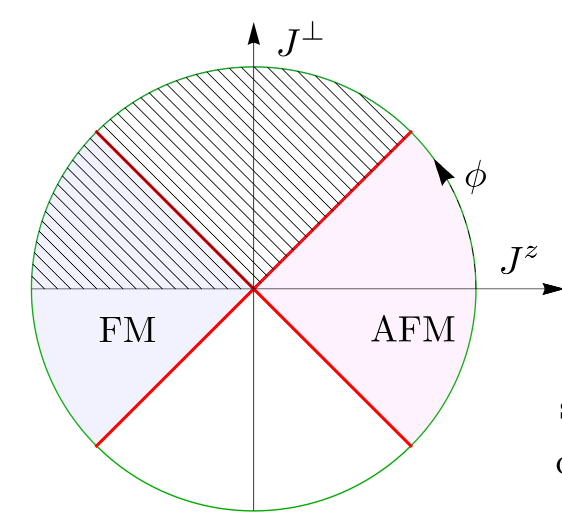
$$P(x_0)\propto e^{-x_0^2}\qquad g=0$$

Two-component bosons

The upper branch of harmonically trapped two-component bosons with near-resonant intra- and interparticle interactions is described by the XXZ model, the coefficients of which may be determined with the fermionic ansatz

$$\mathcal{H}=\sum_{i=1}^{N-1}\eta_i\left[J^\perp(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+J^zS_i^zS_{i+1}^z\right]$$

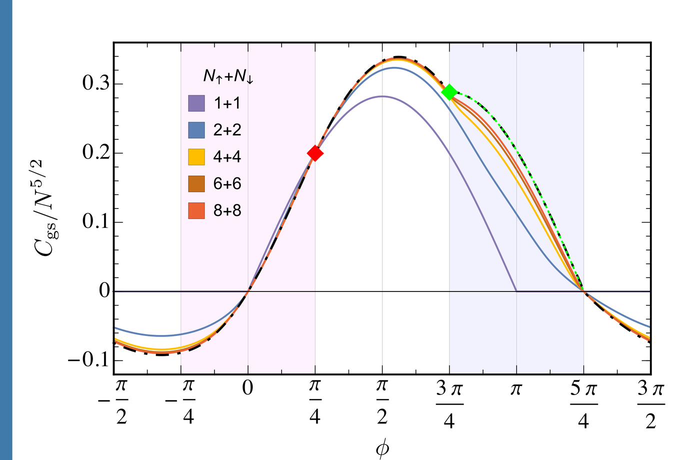
- This has both possible ferromagnetic (FM) and antiferro-magnetic (AFM) ground states



The upper branch ground state. In the hatched region this is the actual ground state.

$$\sin\phi=g/g_{\uparrow\downarrow}\equiv J^\perp$$
$$\cos\phi=g/g_{\uparrow\downarrow}-g/g_{\uparrow\uparrow}\equiv J^z$$

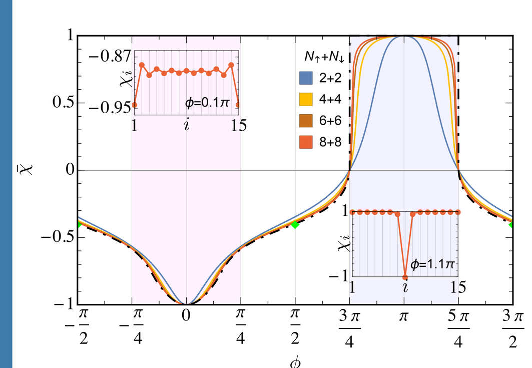
- As opposed to the Fermi gas, one can *adiabatically tune* between the phases - in particular to the analogue of itinerant electron ferromagnetism
- We find clear evidence of a 1st order FM phase transition via emergence of kinks in the energy even for few particles



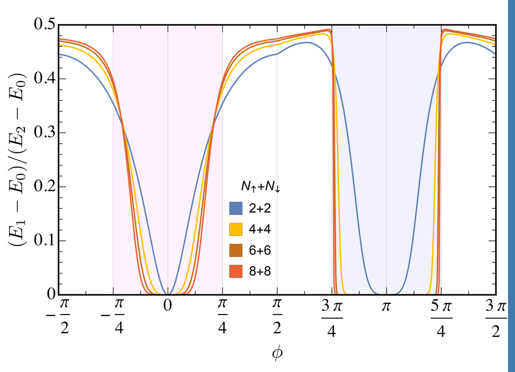
$$E_{gs}=E_0-C_{gs}/g$$

Black: extrapolation to thermodyn. limit
Green: Lieb-Liniger solution (diamond) extended to whole FM region

Trap-averaged magnetic correlation function $\bar\chi=\sum_i\frac{\langle\sigma_i^z\sigma_{i+1}^z\rangle}{N-2}$



Near degeneracy of ground and 1st excited states



Conclusions and outlook

- ***We solved for all wavefunctions in the ground state manifold of a single ↓ particle strongly interacting with N ↑ fermions in an essentially combinatorial manner. Our ansatz has wavefunction overlaps very close to 1.***
- ***The Heisenberg spin chain is valid for any $N_\uparrow$, $N_\downarrow$, and can be extended to 2-component bosons***
- ***The harmonically trapped strongly-interacting Bose gas appears ideally suited for the realisation of quantum magnetic phases.***

References

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